Large N relationship of weakly coupled QCD on the hypersphere with strongly coupled lattice QCD

Joyce Myers

Niels Bohr Institute

based on work with Alexander S. Christensen and Peter D. Pedersen

and on previous work with Timothy J. Hollowood

XQCD 2013

Universität Bern

7 August 2013

Outline

- Introduction: A large N_c relationship of weakly-coupled continuum QCD on $S^1 \times S^3$ with a truncated action, and strongly-coupled lattice QCD with static quarks from a $3d$ effective spin model
- Corrections: What happens to this relationship at the next order in the strong coupling and hopping expansions?
  - The strong coupling expansion to $\mathcal{O}(\beta^{2N_t})$
  - The hopping expansion to $\mathcal{O}(\kappa^{2N_t})$

What is the leading order relationship?

What we're investigating is a large N_c correspondence between equations of motion.

Using this correspondence observables in one theory can be calculated from observables in the other under a suitable transformation of parameters.

QCD on $S^1 \times S^3$ with a truncated action	$\overset{\longleftrightarrow}{N_c \to \infty}$	Lattice QCD 3d effective spin model
$\lambda \to 0$		$\lambda \to \infty$
small volume		large volume
any $m \lesssim \mu$		heavy quarks, $m \lesssim \mu$
continuum		lattice

What can be mapped?

We mapped the polyakov lines, quark number, and resulting phase diagram for QCD with $\mu \neq 0$ from $S^1 \times S^3$ to the lattice strong coupling expansion with heavy quarks.

1-loop action on $S^1 \times S^3$

[Aharony et al - Adv.Theor.Math.Phys. 8 (2004) [hep-th/0310285]]

QCD action of Polyakov lines $\rho_n = \frac{1}{N_c}\sum_{j=1}^{N_c} e^{in\theta_j}$.

$$S_{QCD} = N_c^2 \sum_{n=1}^{\infty} \frac{1}{n}(1 - z_{vn})\,\rho_n \rho_{-n} + N_f N_c \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z_{fn} \left(e^{n\beta\mu}\rho_n + e^{-n\beta\mu}\rho_{-n}\right),$$

- $\beta = 1/T$, $R =$ radius of S^3, $m =$ quark mass

$$z_{vn} = \sum_{\ell=1}^{\infty} d_\ell^{(v)} e^{-n\beta\varepsilon_\ell^{(v)}} = 2\sum_{\ell=1}^{\infty} \ell(\ell+2) e^{-n\beta(\ell+1)/R}$$

$$z_{fn} = \sum_{\ell=1}^{\infty} d_\ell^{(f)} e^{-n\beta\varepsilon_\ell^{(f)}} = 2\sum_{\ell=1}^{\infty} \ell(\ell+1) e^{-n\frac{\beta}{R}\sqrt{(\ell+\frac{1}{2})^2 + m^2R^2}}$$

YM deconfinement transition at $z_{v1} = 1$ ($T_c \simeq 0.759/R$) [Aharony et al (2003)].

Lattice strong coupling expansion

[Langelage, Lottini, Philipsen - JHEP 1102 (2011) 057[arXiv:1010.0951]]

After integrating out the spatial link variables the lattice Yang-Mills partition function can be simplified by using the character expansion

$$Z_{YM} = \int_{SU(N_c)} \prod_z \mathrm{d}W_z \prod_{\langle xy \rangle} \left[1 + \sum_R \lambda_R \left[\chi_R(W_x)\chi_R(W_y^\dagger) + \chi_R(W_x^\dagger)\chi_R(W_y) \right] \right] .$$

$\chi_R(W_x) = \mathrm{Tr}_R(W_x)$ are characters of Polyakov lines $W_{\mathbf{x}} = \prod_{\tau=1}^{N_\tau} U_0(\mathbf{x}, \tau)$ in representation R. The $\prod_{\langle xy \rangle}$ is over nearest neighbor sites.

The λ_R are expansion parameters in powers of $\frac{1}{g^2 N_c}$,

$$\lambda_R = [u_R]^{N_\tau} \left[1 + \mathcal{O}(u) \right] ,$$

with

$$u \equiv u_F \xrightarrow[N_c \to \infty]{} \frac{1}{g^2 N_c} .$$

Hopping expansion - static quark limit

[Langelage, Lottini, Philipsen - JHEP 1102 (2011) 057 [arXiv:1111.4953]]

The fermion determinant can be approximated in the static, heavy quark limit by the hopping expansion

$$\log\det(\not{D} + \gamma_0\mu + m) = a_1 h[e^{\mu/T}\mathrm{Tr}W_x + e^{-\mu/T}\mathrm{Tr}W_x^\dagger] + a_2 h^2[e^{2\mu/T}\mathrm{Tr}(W_x^2) + e^{-2\mu/T}\mathrm{Tr}(W_x^{\dagger 2})] + \dots .$$

For Wilson fermions

$$a_n = 2\frac{(-1)^n}{n}\,, \qquad h = \kappa^{N_t}\left[1 + \mathcal{O}(\kappa^2)\right]\,, \qquad \kappa = \frac{1}{ma + d + 1}\,.$$

See also [De Pietri, Feo, Seiler, Stamatescu - Phys.Rev. D76 (2007) 114501 [arXiv:0705.3420]]

What are the leading order transformations?

[Hollowood and JM - JHEP 1210 (2012) 067 [arXiv:1207.4605]]

To leading order in a combined lattice strong coupling and hopping expansion the action is [Damgaard and Patkós (Phys. Lett. B **172** (1986) 369)]

$$S_{lat}^{(1)} - S_{Vdm} = -2u^{N_t} D \sum_x \left[\langle \mathrm{Tr} W \rangle \mathrm{Tr} W_x^\dagger + \langle \mathrm{Tr} W^\dagger \rangle \mathrm{Tr} W_x - \langle \mathrm{Tr} W \rangle \langle \mathrm{Tr} W^\dagger \rangle \right]$$
$$- 2N_f \kappa^{N_\tau} \sum_x \left[e^{\mu/T} \mathrm{Tr} W_x + e^{-\mu/T} \mathrm{Tr} W_x^\dagger \right] .$$

From 1-loop perturbation theory the action for QCD on $S^1 \times S^3$ is

$$S_{S^1 \times S^3} - S_{Vdm} = -N_c^2 \mathbf{z}_{v1} \rho_1 \rho_{-1}$$
$$- N_f N_c \mathbf{z}_{f1} \left(e^{\mu/T} \rho_1 + e^{-\mu/T} \rho_{-1} \right) ,$$

where the action is truncated at $n = 1$. This is a good approximation for $\mu < \varepsilon_{f1}$ and T not too high (such that $\mathbf{z}_{v1}$, $\mathbf{z}_{f1} e^{\mu\beta} \gg \mathbf{z}_{v2}$, $\mathbf{z}_{f2} e^{2\mu\beta}$).

Transformations:

$$\rho_1 \leftrightarrow \frac{1}{N_c} \langle \mathrm{Tr} W \rangle ,$$
$$\rho_{-1} \leftrightarrow \frac{1}{N_c} \langle \mathrm{Tr} W^\dagger \rangle ,$$

$$\mathbf{z}_{v1} \to 2u^{N_t} D ,$$
$$\mathbf{z}_{f1} \to 2\kappa^{N_t} ,$$

Lattice strong coupling expansion

[Langelage, Lottini, Philipsen - JHEP 1102 (2011) 057[arXiv:1010.0951]]

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$\chi_R(W_x) = \mathrm{Tr}_R(W_x)$ are characters of Polyakov lines $W_\mathbf{x} = \prod_{\tau=1}^{N_\tau} U_0(\mathbf{x}, \tau)$ in representation R. The $\prod_{\langle xy \rangle}$ is over nearest neighbor sites.

The λ_R are expansion parameters in powers of $\frac{1}{g^2 N_c}$,

$$\lambda_R = \left[\boxed{u_R} \right]^{N_\tau} \left[1 + \mathcal{O}(u) \right] ,$$

with

$$u \equiv u_F \xrightarrow[N_c \to \infty]{} \frac{1}{g^2 N_c} .$$

u_R for general N_c

The couplings can be obtained from

$$u_R = \frac{1}{d_R}\frac{\tilde{u}_R}{\tilde{u}_0} ,$$

where d_R is the dimension of the representation R,

$$\tilde{u}_R = \sum_{n=-\infty}^{\infty} \det\left[I_{\lambda_j+i-j+n}(x)\right] ,$$

and

$$\tilde{u}_0 = \sum_{n=-\infty}^{\infty} \det\left[I_{i-j+n}(x)\right] ,$$

with $x \equiv \frac{2}{g^2}$. $I_\nu(x)$ is the modified Bessel function of the first kind. The λ_j represent the Young tableau of the representation R.

Labeling a representation: λ_i

The Young tableau of a representation R is labeled by $(\mu) = (\mu_1, \mu_2, ..., \mu_{N-1})$, where μ_1 is the number of columns with 1 box, μ_2 is the number of columns with 2 boxes, ..., and ending with the number of columns with $N-1$ boxes.

The λ_i descend in magnitude $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_N$. The definition is $\{\lambda\} = \{\lambda_1, \lambda_2, ..., \lambda_N\}$, where $\lambda_i = \mu_i + \mu_{i+1} + ... + \mu_{N-1}$, such that $\lambda_{N-1} = \mu_{N-1}$, and $\lambda_N = 0$.

Double Young diagrams

[Drouffe and Zuber]

In the large N_c limit the couplings can also be obtained from double Young diagrams. The formula for the u_R simplifies to the form

$$u_R = \frac{1}{d_R}\frac{\sigma_{\{m\}}}{|m|!}\frac{\sigma_{\{n\}}}{|n|!}(N_c u)^{|\lambda|}\,,$$

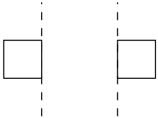
where

$$\frac{\sigma_{\{k\}}}{|k|!} = d_k \prod_{i=0}^{N_c-1} \frac{i!}{(\lambda_{N_c-i}+i)!}\,.$$

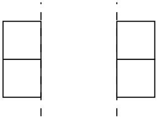
Here the $\lambda_i = \{m_1 m_2, ..., 0, 0, ..., -n_1, -n_2, ...\}$ represent the double Young tableau of the representation R.

Double Young diagrams

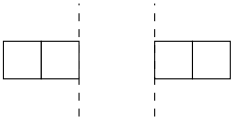
fundamental: $\lambda = \{1\}, \{-1\}$



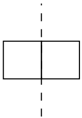
antisymmetric $\lambda = \{1, 1\}, \{-1, -1\}$



symmetric $\lambda = \{2\}, \{-2\}$



adjoint $\lambda = \{1, -1\}$



u_R for $N_c \to \infty$

Fundamental

$$u_F \xrightarrow[N_c \to \infty]{} u \,.$$

Symmetric

$$u_S \xrightarrow[N_c \to \infty]{} u^2 \,.$$

Antisymmetric

$$u_{AS} \xrightarrow[N_c \to \infty]{} u^2 \,.$$

Adjoint

$$u_{Adj} \xrightarrow[N_c \to \infty]{} u^2 \,.$$

Lattice strong coupling expansion

[Langelage, Lottini, Philipsen - JHEP 1102 (2011) 057[arXiv:1010.0951]]

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$\chi_R(W_x) = \mathrm{Tr}_R(W_x)$ are characters of Polyakov lines $W_{\mathbf{x}} = \prod_{\tau=1}^{N_\tau} U_0(\mathbf{x}, \tau)$ in representation R. The $\prod_{\langle xy \rangle}$ is over nearest neighbor sites.

The λ_R are expansion parameters in powers of $\frac{1}{g^2 N_c}$,

$$\lambda_R = [u_R]^{N_\tau} \left[1 + \mathcal{O}(u)\right] ,$$

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Characters

One can obtain the characters $\chi_R(W_x)$ from the Frobenius formula,

$$\chi_R(W) = \mathrm{Tr}_R W = \frac{1}{n!} \sum_{\mathbf{j} \in S_n} \chi_R(\mathbf{j}) (\mathrm{Tr}_F W)^{j_1} (\mathrm{Tr}_F W^2)^{j_2} ... (\mathrm{Tr}_F W^n)^{j_n} ,$$

where

n is the number of boxes in the Young tableau of the representation R,

$\chi_R(\mathbf{j})$ is the group character, in the representation R, of the permutations $\mathbf{j} = j_1, j_2, ..., j_n$, of the symmetric group S_n.

In practice it is simpler to obtain the characters for the symmetric representations,

$$\chi_S(\mathbf{j}) = \frac{n!}{\prod_{k=1}^{n} k^{j_k} j_k!} ,$$

then apply suitable tensor product decompositions to obtain the other characters.

Characters

Symmetric representation is:

$$\mathrm{Tr}_S W = \mathrm{Tr}_{(2,0,\ldots,0)} W = \frac{1}{2}\left[(\mathrm{Tr}\, W)^2 + (\mathrm{Tr}\, W^2)\right] \ ,$$

From $(1,0,...,0) \otimes (1,0,...,0) = (0,2,0,...,0) \oplus (2,0,...,0)$, the antisymmetric representation is:

$$\mathrm{Tr}_{AS} W = \mathrm{Tr}_{(0,2,0,\ldots,0)} W = \frac{1}{2}\left[(\mathrm{Tr}\, W)^2 - (\mathrm{Tr}\, W^2)\right] \ ,$$

From $(1,0,...,0) \otimes (0,...,0,1) = (1,0,...,0,1) \oplus \mathbf{1}$ the adjoint representation is:

$$\mathrm{Tr}_{Adj} W = \mathrm{Tr}_{(1,0,\ldots,0,1)} W = \mathrm{Tr}\, W\, \mathrm{Tr}\, W^\dagger - 1 \, .$$

Corrections to the action

Adding up the contributions from the fundamental, symmetric, antisymmetric, and adjoint representations, the correction to the action at $\mathcal{O}(u^{2N_t})$ is

$$\lambda_2 S_g^{(2)} = -\frac{u^{2N_t}}{2} \sum_{\langle xy \rangle} \Big[\mathrm{Tr}(W_x^2)\mathrm{Tr}(W_y^{\dagger 2}) + \mathrm{Tr}(W_x^{\dagger 2})\mathrm{Tr}(W_y^2) - \mathrm{Tr}W_x \mathrm{Tr}W_x^\dagger - \mathrm{Tr}W_y \mathrm{Tr}W_y^\dagger \Big] .$$

Using large N_c factorization this can be rewritten as

$$\lambda_2 S_g^{(2)} = -d \; u^{2N_t} \sum_{x} \Big[\langle \mathrm{Tr}(W^2) \rangle \mathrm{Tr}(W_x^{\dagger 2}) + \langle \mathrm{Tr}(W^{\dagger 2}) \rangle \mathrm{Tr}(W_x^2) - \langle \mathrm{Tr}(W^{\dagger 2}) \rangle \langle \mathrm{Tr}(W^2) \rangle - 2 \; \mathrm{Tr}W_x \mathrm{Tr}W_x^\dagger \Big] .$$

Lattice strong coupling expansion

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Decorations

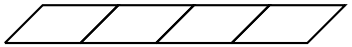
[Langelage, Lottini, Philipsen - JHEP 1102 (2011) 057 [arXiv:1010.0951]

There are higher order corrections to nearest neighbor contribution to the action from fundamental representation Polyakov lines. These are corrections to the $\mathcal{O}(u^{N_\tau})$ terms which take the form

$$u^{N_\tau} \boxed{\lambda'_{g1}} S_g^{(1)} = u^{N_\tau} \boxed{\lambda'_{g1}(u, N_\tau)} \sum_{\langle xy \rangle} \left[\mathrm{Tr} W_x \mathrm{Tr} W_y^\dagger + \mathrm{Tr} W_x^\dagger \mathrm{Tr} W_y \right]$$

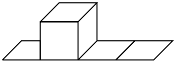
Some contributions are:

No decorations



$$u^{N_\tau} S_g^{(1)}$$

One raised plaquette decoration (3 spatial dimensions)

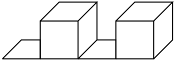


$$u^{N_\tau} \left[4 N_\tau u^4 \right] S_g^{(1)}$$

Decorations

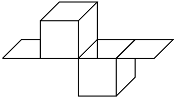
[Langelage, Lottini, Philipsen - JHEP 1102 (2011) 057 [arXiv:1010.0951]

Two raised plaquette decorations which are not next to each other



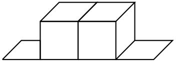
$$u^{N_\tau}\left[\frac{1}{2!}(4N_\tau u^4)\cdot 4(N_\tau-3)u^4\right]S_g^{(1)}$$

Two consecutive raised plaquette decorations which do not face the same direction



$$u^{N_\tau}\left[4N_\tau u^4\cdot 3u^4\right]S_g^{(1)}$$

Two consecutive attached raised plaquette decorations



$$u^{N_\tau}\left[4N_\tau u^6\right]S_g^{(1)}$$

or 3, 4, etc.

Hopping expansion - static quark limit

[Langelage, Lottini, Philipsen - JHEP 1102 (2011) 057 [arXiv:1111.4953]]

The fermion determinant can be approximated in the static, heavy quark limit by the hopping expansion

$$\log\det(\not{D} + \gamma_0\mu + m) = a_1 h[e^{\mu/T}\mathrm{Tr}\,W_x + e^{-\mu/T}\mathrm{Tr}\,W_x^\dagger] + a_2 h^2[e^{2\mu/T}\mathrm{Tr}(W_x^2) + e^{-2\mu/T}\mathrm{Tr}(W_x^{\dagger 2})] + \dots .$$

For Wilson fermions

$$a_n = 2\frac{(-1)^n}{n}, \qquad h = \kappa^{N_t}\left[1 + \boxed{\mathcal{O}(\kappa^2)}\right], \qquad \kappa = \frac{1}{ma + d + 1}.$$

See also [De Pietri, Feo, Seiler, Stamatescu - Phys.Rev. D76 (2007) 114501 [arXiv:0705.3420]]

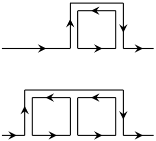
Hopping expansion - corrections

[Langelage, Lottini, Philipsen - JHEP 1102 (2011) 057 [arXiv:1111.4953]]

There are also corrections to the hopping expansion in the 'static' quark limit which include short spatial detours. These corrections to the leading $\mathcal{O}(\kappa^{N_\tau})$ terms contribute before the $\mathcal{O}(\kappa^{2N_\tau})$ contributions.

$$\kappa^{N_\tau} \lambda'_{f1} S_f^{(1)} = 2\kappa^{N_\tau} \lambda'_{f1}(\kappa, u, N_\tau)[e^{\mu/T} \mathrm{Tr} W_x + e^{-\mu/T} \mathrm{Tr} W_x^\dagger]$$

For example:



$$\kappa^{N_\tau} \left[6 N_\tau \kappa^2 \sum_{l=1}^{N_\tau - 1} u^l \right] S_f^{(1)} \, .$$

...

What is the new correspondence?

$$S_{lat} = -2u^{N_t} \boxed{\lambda'_{g1}} D \sum_x \left[\langle \mathrm{Tr} W \rangle \mathrm{Tr} W_x^\dagger + \langle \mathrm{Tr} W^\dagger \rangle \mathrm{Tr} W_x - \langle \mathrm{Tr} W \rangle \langle \mathrm{Tr} W^\dagger \rangle \right]$$

$$- d\ u^{2N_t} \sum_x \left[\boxed{\langle \mathrm{Tr}(W^2) \rangle \mathrm{Tr}(W_x^{\dagger 2}) + \langle \mathrm{Tr}(W^{\dagger 2}) \rangle \mathrm{Tr}(W_x^2)} \right.$$

$$\left. \boxed{-\langle \mathrm{Tr}(W^{\dagger 2}) \rangle \langle \mathrm{Tr}(W^2) \rangle - 2\ \mathrm{Tr} W_x \mathrm{Tr} W_x^\dagger} \right]$$

$$+ N_f \sum_x \left[-2\kappa^{N_\tau} \boxed{\lambda'_{f1}} \left[e^{\mu/T} \mathrm{Tr} W_x + e^{-\mu/T} \mathrm{Tr} W_x^\dagger \right] \right.$$

$$\left. \boxed{+\kappa^{2N_\tau} \left[e^{2\mu/T} \mathrm{Tr}(W_x^2) + e^{-2\mu/T} \mathrm{Tr}(W_x^{\dagger 2}) \right]} \right] .$$

$$S_{S^1 \times S^3} - S_{Vdm} = -N_c^2 \left[\mathbf{z}_{v1} \rho_1 \rho_{-1} \boxed{+\frac{1}{2} \mathbf{z}_{v2} \rho_2 \rho_{-2}} \right]$$

$$+ N_f N_c \left[-\mathbf{z}_{f1} \left(e^{\mu/T} \rho_1 + e^{-\mu/T} \rho_{-1} \right) \boxed{+\frac{1}{2} \mathbf{z}_{f2} \left(e^{2\mu/T} \rho_2 + e^{-2\mu/T} \rho_{-2} \right)} \right] .$$

Transformations

$$\rho_1 \leftrightarrow \frac{1}{N_c}\langle \mathrm{Tr} W\rangle\,, \qquad \rho_2 \leftrightarrow \frac{1}{N_c}\langle \mathrm{Tr}(W^2)\rangle\,,$$

$$\rho_{-1} \leftrightarrow \frac{1}{N_c}\langle \mathrm{Tr} W^\dagger\rangle\,, \qquad \rho_{-2} \leftrightarrow \frac{1}{N_c}\langle \mathrm{Tr}(W^{\dagger 2})\rangle\,,$$

$$\mathbf{z}_{v1} \rightarrow 2u^{N_t} D\lambda'_{g1}\,, \qquad \mathbf{z}_{v2} \rightarrow u^{2N_t} D\,,$$

$$\mathbf{z}_{f1} \rightarrow 2\kappa^{N_t}\lambda'_{f1}\,, \qquad \mathbf{z}_{f2} \rightarrow 2\kappa^{2N_t}\,,$$

Conclusions

- A large N_c relationship of EOMs in QCD on $S^1 \times S^3$ from 1-loop perturbation theory, and lattice QCD from a combined strong coupling and hopping expansion, can still be defined when the actions are truncated at 2 windings of the Polyakov lines.
- For lattice variables β and κ it is unclear how to get to the theory on $S^1 \times S^3$, but one can still go from $S^1 \times S^3$ to the lattice theory.
- The lattice action, and the representation-dependent couplings, take a simplified form up to $\mathcal{O}(\beta^{2N_\tau})$ and $\mathcal{O}(h^2)$, in the large N_c limit.

Thanks!

Special thanks to Christian Holm Christensen for giving us a copy of his master thesis and to David Gross and Jens Langelage for several insightful discussions.