Chiral Symmetry and $U(1)_A$ Symmetry in Finite Temperature QCD with Domain-Wall Fermion

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Outline

- Introduction
- Observables for probing the restoration of $SU(2)_L \times SU(2)_R$ and $U(1)_A$
- Preliminary Results
- Concluding Remarks

Introduction

In QCD, the classical action of N_f massless quarks has the symmetry

$$SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A$$

In the quantum theory, at zero temperature, the chiral symmetry $SU(N_f)_L \times SU(N_f)_R$ is broken spontaneously to $SU(N_f)_V$ by the vacuum of QCD, and the $U(1)_A$ is broken by the axial anomaly.

It is expected that at high temperature, both chiral symmetry and $U(1)_A$ are restored. The question is, at what temperature T_c the chiral symmetry is restored, and whether $U(1)_A$ is also restored at $T_1 \approx T_c$

Lattice QCD with exact chiral symmetry is in a good position to answer these questions. HotQCD (DWF with not so small residual mass)

JLQCD (overlap fermion with fixed topology)

Domain-Wall Fermion

$$A_{\text{dwf}} = \sum_{s,s'=1}^{N_s} \sum_{x,x'} \bar{\psi}_{x,s} \left[(I + \rho_s D_w)_{x,x'} \delta_{s,s'} - (I - \sigma_s D_w)_{x,x'} (P_- \delta_{s',s+1} + P_+ \delta_{s',s-1}) \right] \psi_{x',s'}$$

$$\equiv \bar{\Psi} D_{\text{dwf}} \Psi$$

$$D_w = \sum_{\mu=1}^{4} \gamma_\mu t_\mu + W - m_0, \qquad m_0 \in (0,2)$$

$$\rho_s = c\omega_s + d$$
$$\sigma_s = c\omega_s - d$$
$$c, d \ \text{(constants)}$$

$$t_\mu(x,x') = \frac{1}{2}\left[U_\mu(x) \delta_{x',x+\mu} - U_\mu^\dagger(x') \delta_{x',x-\mu} \right]$$

$$W(x,x') = \sum_{\mu=1}^{4} \frac{1}{2}\left[2\delta_{x,x'} - U_\mu(x) \delta_{x',x+\mu} - U_\mu^\dagger(x') \delta_{x',x-\mu} \right]$$

with boundary conditions

$$P_+ \psi(x,0) = -r m_q P_+ \psi(x,N_s), \quad m_q\text{: bare mass}, \quad r = 1/[2m_0(1-dm_0)]$$

$$P_- \psi(x,N_s+1) = -r m_q P_- \psi(x,1), \qquad P_\pm = \frac{1}{2}(1 \pm \gamma_5)$$

Domain-Wall Fermion (cont.)

The action for Pauli-Villars fields is

$$A_{PV} = \sum_{s,s'=1}^{N_s} \sum_{x,x'} \bar{\phi}_{x,s} \left[\left(I + \rho_s D_w \right)_{x,x'} \delta_{s,s'} - \left(I - \sigma_s D_w \right)_{x,x'} \left(P_- \delta_{s',s+1} + P_+ \delta_{s',s-1} \right) \right] \phi_{x',s'}$$

with boundary conditions:

$$P_+ \phi(x,0) = -P_+ \phi(x, N_s),$$

$$P_- \phi(x, N_s + 1) = -P_- \phi(x,1)$$

$$\int [d\bar{\psi}][d\psi][d\bar{\phi}][d\phi] \exp\left(-A_{\mathrm{dwf}} - A_{\mathrm{PV}}\right) = \det D(m_q)$$

The effective 4D Dirac operator

$$D(m_q) = m_q + \left(m_0 (1 - d m_0) - \frac{m_q}{2} \right) \left[1 + \gamma_5 S(H) \right], \quad H = \frac{c H_w}{1 + d \gamma_5 H_w}$$

$$\lim_{N_s \to \infty} S(H) = \frac{H}{\sqrt{H^2}}$$

Domain-Wall Fermion (cont.)

- Conventional DWF with Shamir kernel

$$\left[c = d = 1/2,\ \omega_s = 1 \Rightarrow \rho_s = c\omega_s + d = 1,\ \sigma_s = c\omega_s - d = 0\right]$$

$$D(m_q) = m_q + \left(\frac{m_0}{2}(2 - m_0) - \frac{m_q}{2}\right)\left[1 + \gamma_5 S_{\text{polar}}(H)\right], \quad H = \frac{H_w}{2 + \gamma_5 H_w}$$

$$S_{\text{polar}}(H) = \frac{1 - T^{N_s}}{1 + T^{N_s}}, \qquad T = \frac{1 - H}{1 + H}$$

$$= \begin{cases} H\left(\dfrac{2}{N_s}\displaystyle\sum_{l=1}^{n}\dfrac{b_l}{H^2 + d_l}\right), & N_s = 2n \\ H\left(\dfrac{1}{N_s} + \dfrac{2}{N_s}\displaystyle\sum_{l=1}^{n}\dfrac{b_l}{H^2 + d_l}\right), & N_s = 2n + 1 \end{cases}$$

$$b_l = \sec^2\left[\frac{\pi}{N_s}\left(l - \frac{1}{2}\right)\right]$$

$$d_l = \tan^2\left[\frac{\pi}{N_s}\left(l - \frac{1}{2}\right)\right]$$

polar approximation of $\dfrac{1}{\sqrt{H^2}}$

Domain-Wall Fermion (cont.)

- Optimal DWF [TWC, Phys. Rev. Lett. 90 (2003) 071601]

$$\omega_s = \frac{1}{\lambda_{\min}} \sqrt{1-\kappa'^2 sn^2\left(v_s;\kappa'\right)}, \quad s = 1,\cdots,N_s$$

where $sn\left(v_s;\kappa'\right)$ is the Jacobian elliptic function with argument v_s

and modulus $\kappa' = \sqrt{1-\lambda_{\min}^2/\lambda_{\max}^2}$, and $\lambda_{\min}^2$ and $\lambda_{\max}^2$ are

lower and upper "bounds" of the eigenvalues of H^2

Then the effective 4D Dirac operator becomes

$$D(m_q) = m_q + \left(m_0(1-dm_0) - \frac{m_q}{2} \right)\left[1+\gamma_5 S_{opt}\left(H\right)\right], \quad H = \frac{cH_w}{1+d\gamma_5 H_w}$$

Domain-Wall Fermion (cont.)

$$S_{opt}(H) = \frac{1-\prod_{s=1}^{N_s} T_s}{1+\prod_{s=1}^{N_s} T_s}, \quad T_s = \frac{1-\omega_s H}{1+\omega_s H}$$

$$= \begin{cases} H R_Z^{(n-1,n)}\left(H^2\right), & N_s = 2n \\ H R_Z^{(n,n)}\left(H^2\right), & N_s = 2n+1 \end{cases}$$

⇧

Zolotarev optimal rational approximation of $\frac{1}{\sqrt{H^2}}$

Chiral Sym Breaking due to Finite Ns

[Y.C. Chen, TWC, Phys. Rev. D 86, 094508 (2012)]

It can be measured by the residual mass

$$m_{res}(y) = \left\langle \frac{\sum_x \langle J_5(x,n)\overline{q}(y)\gamma_5 q(y)\rangle}{\sum_x \langle \overline{q}(x)\gamma_5 q(x)\overline{q}(y)\gamma_5 q(y)\rangle} \right\rangle_{\{U\}}, \quad n = \frac{N_s}{2}$$

$$= \left\langle \frac{\mathrm{Re}\,\mathrm{tr}\left(D_c + m_q\right)^{-1}_{y,y}}{\mathrm{tr}\left[\left(D_c^\dagger + m_q\right)\left(D_c + m_q\right)\right]^{-1}_{y,y}} \right\rangle_{\{U\}} - m_q$$

$$J_5(x,n) \equiv \overline{\psi}_{n+1}(x) P_+ \psi_n(x) - \overline{\psi}_n(x) P_- \psi_{n+1}(x)$$

$\left(D_c + m_q\right)^{-1}$ valence quark propagator with $m_q = m_{sea}$

Chiral Sym Breaking due to Finite Ns (cont)

[Y.C. Chen, TWC, Phys. Rev. D 86, 094508 (2012)]

For lattice QCD with ODWF, it can be shown that

$$M_{\text{res}} \le \frac{d_Z}{2r}\left[\frac{2(1+m)}{2-(3-m)d_Z}\right] \qquad m \equiv r m_q$$

For ODWF, $d_Z \ll 1$ in most cases, and it gives

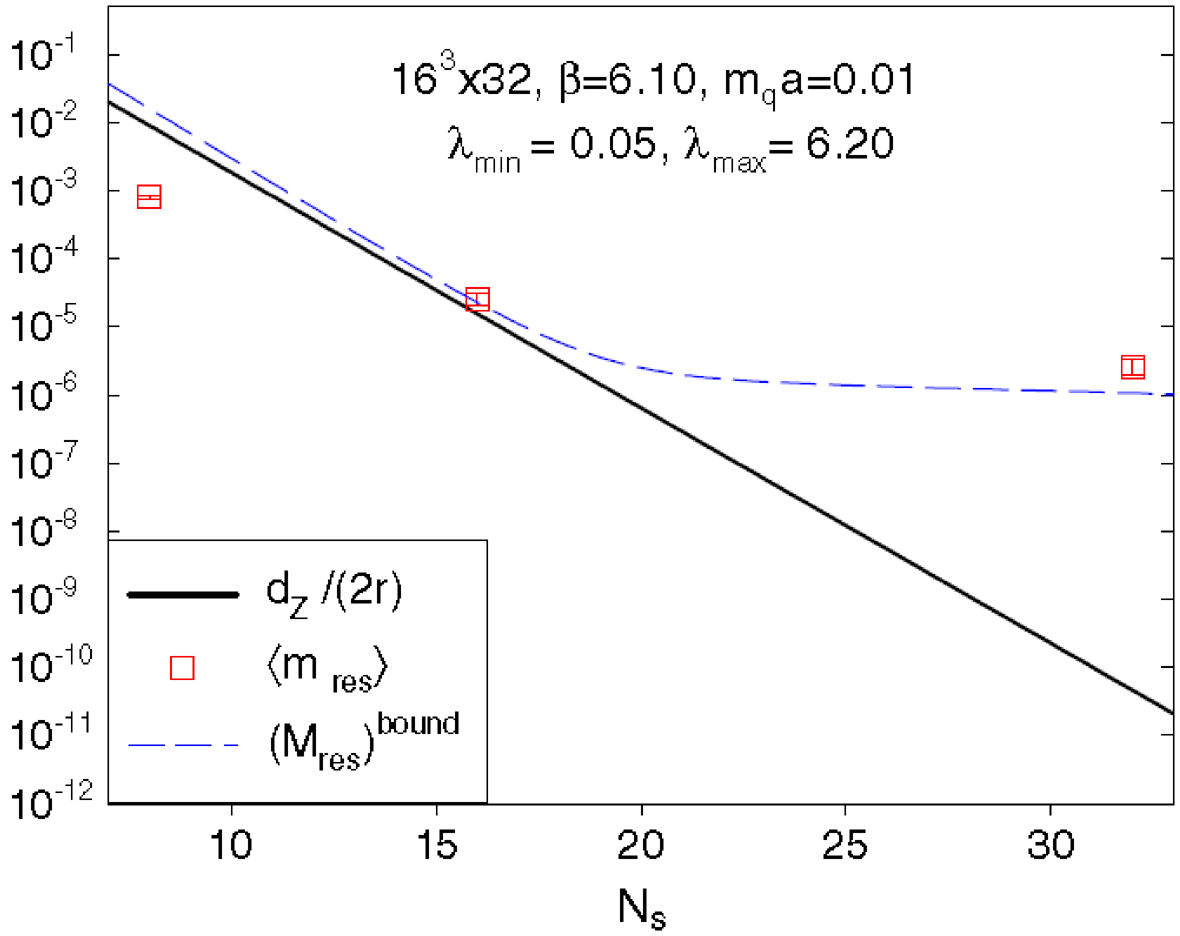
$$M_{\text{res}} \le \frac{d_Z}{2r}(1 + r m_q) \simeq \frac{d_Z}{2r}$$

If there are some eigenvalues of H^2 smaller than $\lambda_{\min}^2$

$$M_{\text{res}} \le \left[\frac{d_Z + (d_a - d_Z)Q_a}{2r}\right]\left[\frac{2(1+m)}{2-(3-m)d_a}\right] \equiv F(N_s, m, N_a/N, h_1),$$

Chiral Sym Breaking due to Finite Ns (cont)

[Y.C. Chen, TWC, Phys. Rev. D 86, 094508 (2012)]



TWQCD simulations of Nf=2 QCD

- Chiral symmetry is preserved to a good precision with the optimal DWF [TWC, Phys. Rev. Lett., 2003].
- Topological sectors are sampled ergodically.
- Multiple-time scale integration and mass preconditioning.
- Even-Odd Preconditioning for the 4D Wilson-Dirac Matrix.
- Omelyan Integrator for the Molecular Dynamics.
- Conjugate gradient with mixed precision.
- Use a GPU cluster of 320 GPUs, with sustained 100 Tflops

Gauge Ensembles of 2-flavor QCD with ODWF

- Lattice size: $16^3 \times 6 \times 16$
- Quark action: Optimal Domain-Wall Fermion (ODWF)
- Gluon action: Wilson plaquette (β = 5.86, 5.87, …, 5.95)
- For each β, three sea-quark masses, $m_q a$ = 0.01, 0.02, 0.03
- For each (β, m), after thermalization, 3000-5000 trajectories have been accumulated. Sampling one configuration every 10 trajectories gives 300-500 confs.
- For each conf, **zero modes** plus **200+200 conjugate pairs of low-lying eigenmodes** of the overlap operator are projected.

Statistics of HMC Trajectories for $16^3 \times 6$

Beta	m=0.01	m=0.02	m=0.03
5.86	2050	1240+	1757+
5.87	2509	2675	2693
5.88	5106	2518+	2363+
5.89	4522+	4848	3233
5.90	3235	2677	1835+
5.91	9216	2193	2267
5.92	3229	2124	1873
5.93	12998	1844	2005
5.94	2235	1797	1900
5.95	5648	2428	7252

This covers $T \sim 140 - 300$ MeV

Residual masses with ODWF at Ns=16

beta	m=0.01	m=0.02	m=0.03
5.86			
5.87			
5.88	0.00059(8)	0.00047(8)	0.00077(5)
5.89	0.00050(7)	0.00035(4)	0.00042(5)
5.90	0.00018(2)	0.00016(4)	0.00022(4)
5.91	0.00020(2)	0.00025(3)	0.00019(3)
5.92	0.00019(4)	0.00018(3)	0.00014(3)
5.93	0.00019(3)	0.00011(2)	0.00011(2)
5.94	0.00013(2)	0.00011(3)	0.00008(1)
5.95	0.00006(2)	0.00006(4)	0.00006(5)

Observables for probing the restoration of $SU(2)_L \times SU(2)_R$ and $U(1)_A$

- Chiral susceptibilities $(\chi_\pi\ ,\ \chi_\delta\ , \chi_\eta\ , \chi_\sigma)$
- Eigenvalue density of the overlap operator

$$\chi_\pi - \chi_\delta = \int_0^\infty d\lambda \frac{4m_q^2\rho(\lambda)}{(m_q^2+\lambda^2)^2}$$

Chiral Susceptibilities

Scalar mesons

$$\delta = \bar{u}d,\ \bar{d}u,\ \frac{1}{\sqrt{2}}\left(\bar{u}u - \bar{d}d\right), \quad \text{flavor non-singlet}$$

$$\sigma = \frac{1}{\sqrt{2}}\left(\bar{u}u + \bar{d}d\right), \quad \text{flavor singlet}$$

$$C_\delta(x) = \left\langle (\bar{u}d)^\dagger(x)(\bar{u}d)(0) \right\rangle$$

$$C_\sigma(x) = \left\langle \sigma^\dagger(x)\sigma(0) \right\rangle$$

$$\chi_\delta = \sum_x C_\delta(x) = -\frac{1}{L_x^3 L_t}\left\langle \mathrm{Tr}(D_c + m_q)^{-2} \right\rangle$$

$$\chi_\sigma = \sum_x C_\sigma(x) = \chi_\delta + \frac{2}{L_x^3 L_t}\left\{\left\langle \left[\mathrm{Tr}(D_c + m_q)^{-1}\right]^2 \right\rangle - \left\langle \mathrm{Tr}(D_c + m_q)^{-1} \right\rangle^2 \right\}$$

Chiral Susceptibilities (cont)

Pseudoscalar mesons

$$\pi = \bar{u}\gamma_5 d,\ \bar{d}\gamma_5 u,\ \frac{1}{\sqrt{2}}\left(\bar{u}\gamma_5 u - \bar{d}\gamma_5 d\right),$$ flavor non-singlet

$$\eta = \frac{1}{\sqrt{2}}\left(\bar{u}\gamma_5 u + \bar{d}\gamma_5 d\right),$$ flavor singlet

$$C_\pi(x) = \left\langle \left(\bar{u}\gamma_5 d\right)^\dagger (x) \left(\bar{u}\gamma_5 d\right)(0) \right\rangle$$

$$C_\eta(x) = \left\langle \eta^\dagger(x)\eta(0) \right\rangle$$

$$\chi_\pi = \sum_x C_\pi(x) = \frac{1}{L_x{}^3 L_t} \left\langle \mathrm{Tr}[\gamma_5(D_c + m_q)]^{-2} \right\rangle$$

$$\chi_\eta = \sum_x C_\eta(x) = \chi_\pi - \frac{2}{L_x{}^3 L_t} \left\{ \left\langle \left[\mathrm{Tr}\gamma_5(D_c + m_q)^{-1}\right]^2 \right\rangle - \left\langle \mathrm{Tr}\gamma_5(D_c + m_q)^{-1} \right\rangle^2 \right\}$$

Chiral Susceptibilities (cont)

$$C_\delta(x) = \left\langle \left(\bar{u}d\right)^\dagger(x)\left(\bar{u}d\right)(0) \right\rangle \qquad C_\pi(x) = \left\langle \left(\bar{u}\gamma_5 d\right)^\dagger(x)\left(\bar{u}\gamma_5 d\right)(0) \right\rangle$$

$$C_\sigma(x) = \left\langle \sigma^\dagger(x)\sigma(0) \right\rangle \qquad C_\eta(x) = \left\langle \eta^\dagger(x)\eta(0) \right\rangle$$

$$\chi_\delta = \sum_x C_\delta(x) \qquad \chi_\pi = \sum_x C_\pi(x)$$

$$\chi_\sigma = \sum_x C_\sigma(x) \qquad \chi_\eta = \sum_x C_\eta(x)$$

$$\left.\begin{array}{l} \chi_\pi = \chi_\sigma \\ \chi_\eta = \chi_\delta \end{array}\right\} \Leftrightarrow \text{restoration of } SU(2)_L \times SU(2)_R$$

$$\left.\begin{array}{l} \chi_\pi = \chi_\delta \\ \chi_\sigma = \chi_\eta \end{array}\right\} \Leftrightarrow \text{restoration of } U(1)_A$$

Chiral Susceptibilities (cont)

$$\chi_\eta = \chi_\pi - \frac{2}{L_x^3 L_t}\left\{\left\langle\left[\mathrm{Tr}\gamma_5 (D_c + m_q)^{-1}\right]^2\right\rangle - \left\langle \mathrm{Tr}\gamma_5 (D_c + m_q)^{-1}\right\rangle^2\right\}$$

$$= \chi_\pi - \frac{2\chi_t}{m_q^2}, \qquad \chi_t = \frac{1}{L_x^3 L_t}\left\{\left\langle Q_{top}^2\right\rangle - \left\langle Q_{top}\right\rangle^2\right\},$$ topological susceptibility

At temperature $T \geq T_c$

$$\left.\begin{aligned} \chi_\pi &= \chi_\sigma \\ \chi_\eta &= \chi_\delta \end{aligned}\right\} \Leftrightarrow \text{restoration of } SU(2)_L \times SU(2)_R$$

$$\chi_\pi - \chi_\eta = \frac{2\chi_t}{m_q^2} \quad \Leftrightarrow \quad \chi_{disc} \equiv \chi_\sigma - \chi_\delta = \frac{2\chi_t}{m_q^2}$$

Chiral Susceptibilities (cont)

At temperature $T \geq T_1$

$$\left.\begin{array}{l} \chi_\pi = \chi_\delta \\ \chi_\sigma = \chi_\eta \end{array}\right\} \Leftrightarrow \text{restoration of } U(1)_A$$

If $T_1 = T_c$, then $\chi_\pi = \chi_\delta = \chi_\sigma = \chi_\eta$ for $T \geq T_c$,
restoration of $SU(2)_L \times SU(2)_R \times U(1)_A$

$$\left(\frac{\chi_t}{m_q^2}\right) \xrightarrow{m_q \to 0} \begin{cases} = 0 \, , & T \geq T_c \\ \propto \dfrac{1}{m_q} \, , & T < T_c \end{cases}$$

Chiral Susceptibilities (cont)

If $T_1 > T_c$, there exists a window $T_c \le T \le T_1$

$$\begin{Bmatrix} \chi_\pi = \chi_\sigma \\ \chi_\eta = \chi_\delta \end{Bmatrix} \text{ but } \begin{Bmatrix} \chi_\pi \neq \chi_\delta \\ \chi_\sigma \neq \chi_\eta \end{Bmatrix}$$

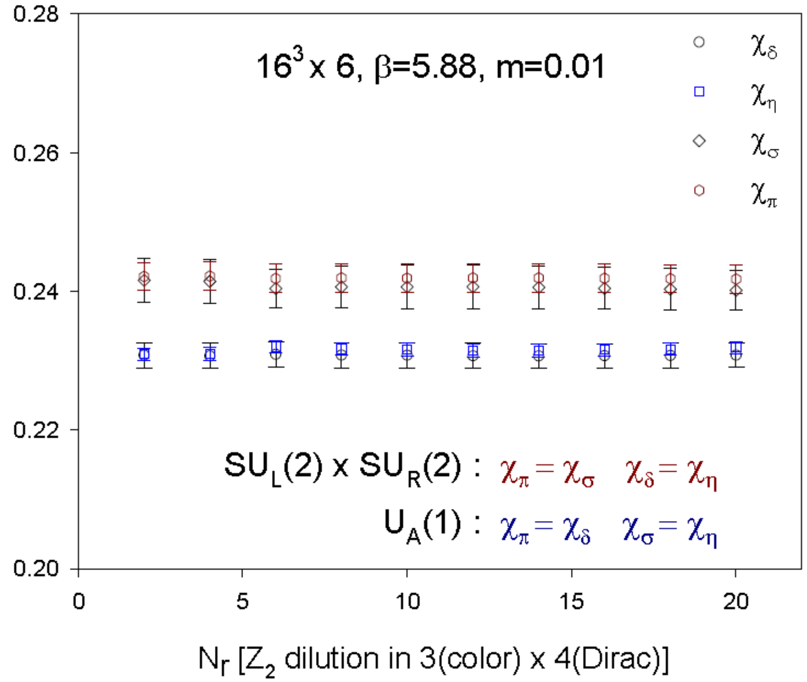
If the chiral symmetry restoration (phase transition) belongs to the $O(4)$ universality class, then we expect

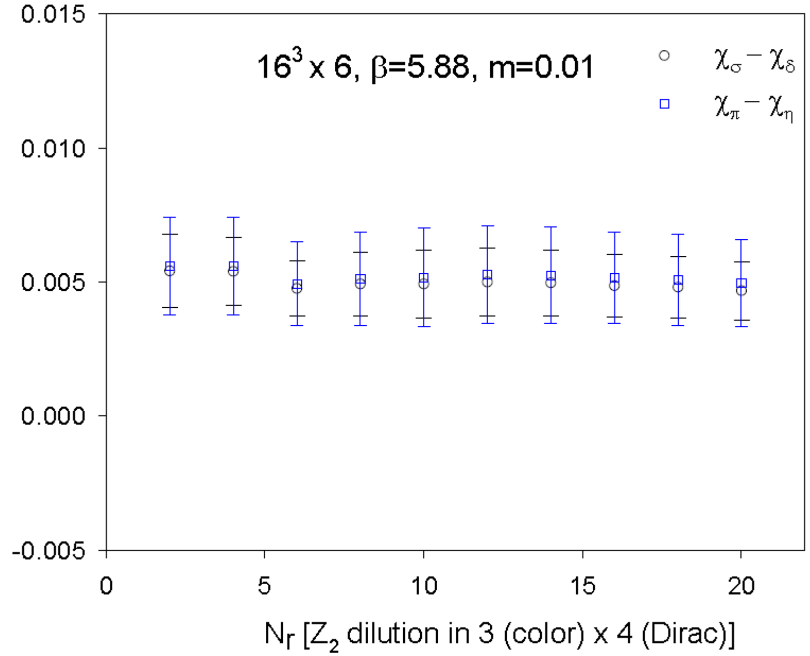
$$\left(\frac{\chi_t}{m_q^2} \right) \xrightarrow{m_q \to 0} \sim (T - T_c)^{-\gamma}, \quad T \ge T_c$$

$$\gamma = 1.453$$

Preliminary Results of Chiral Susceptibilities

The chiral susceptibilities are evaluated with the all-to-all quark propagators, which are computed using 240 Z_2 noises with dilution in the color and the Dirac indices, for each conf. The results are well saturated by the number of noise vectors.

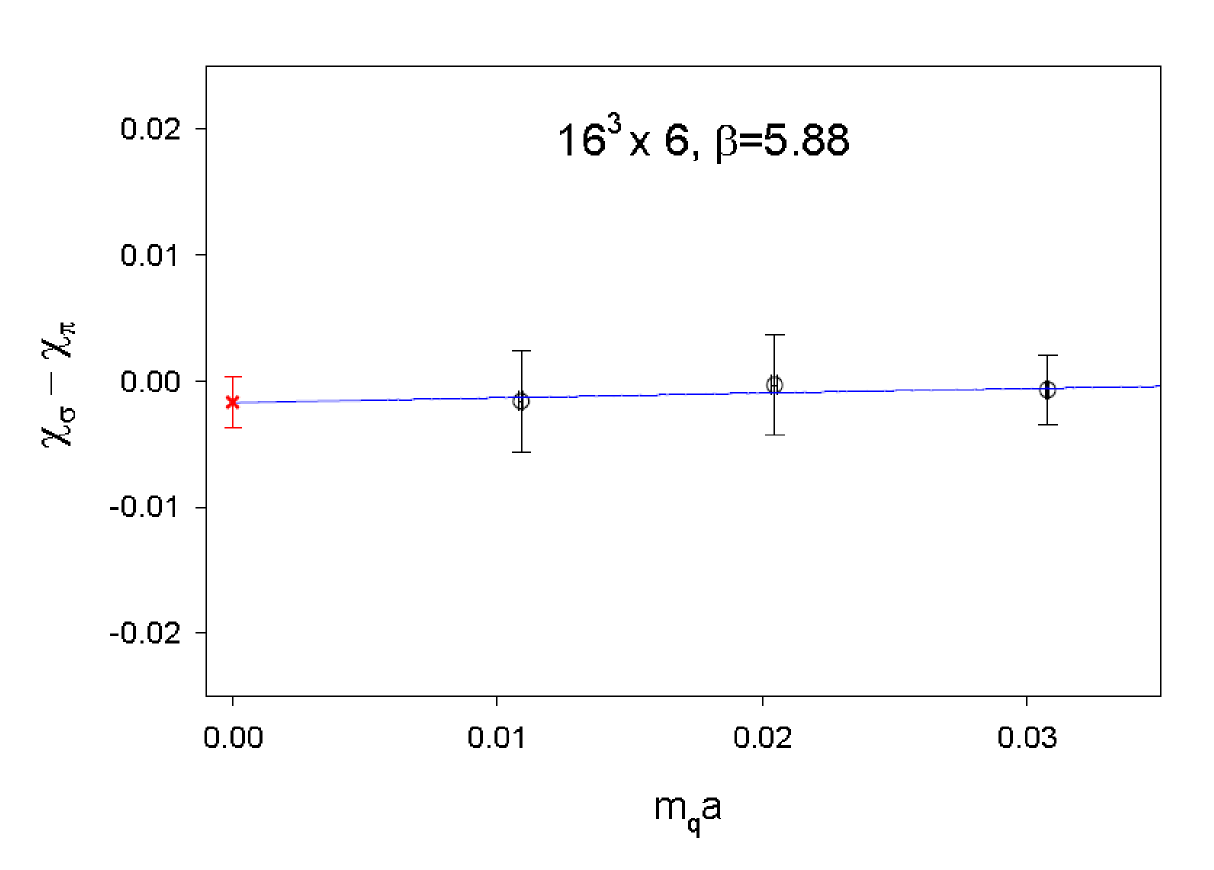




Chiral Susceptibilities (cont.)

$$\chi_\pi \simeq \chi_\sigma \simeq 0.24$$

$T \approx 180$ MeV

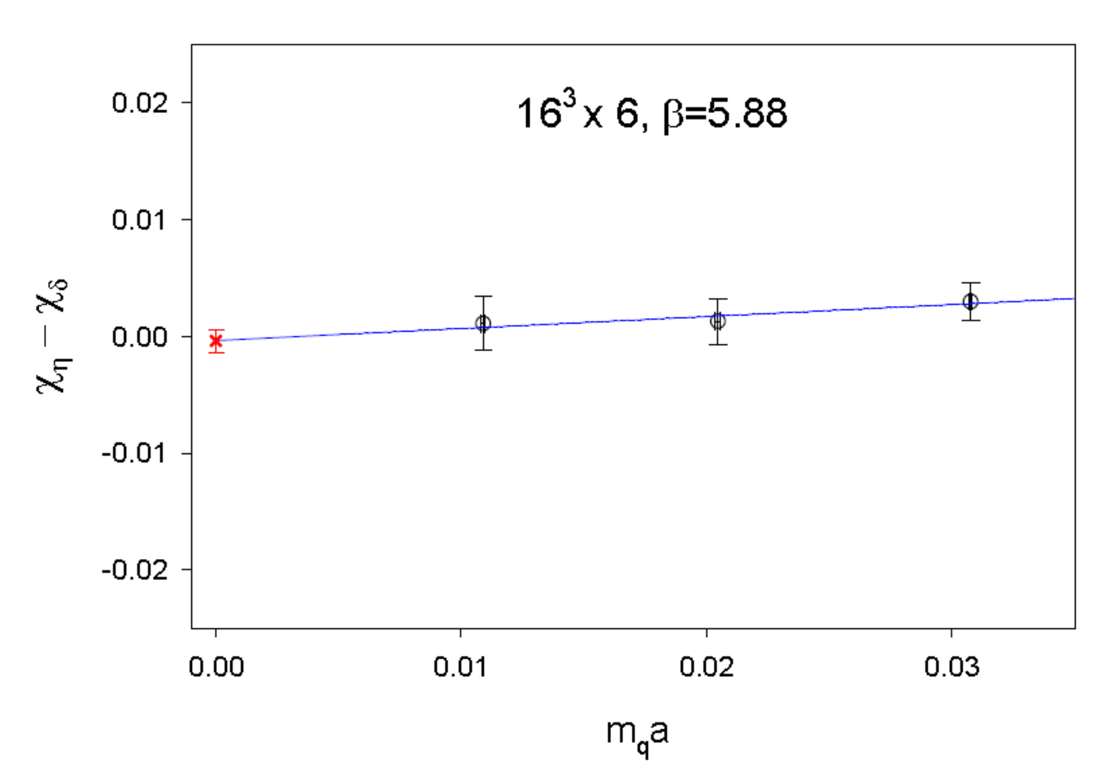


$$\chi_\pi = \chi_\sigma \Rightarrow \text{ restoration of } SU(2)_L \times SU(2)_R$$

Chiral Susceptibilities (cont.)

$$\chi_\eta \simeq \chi_\delta \simeq 0.23$$

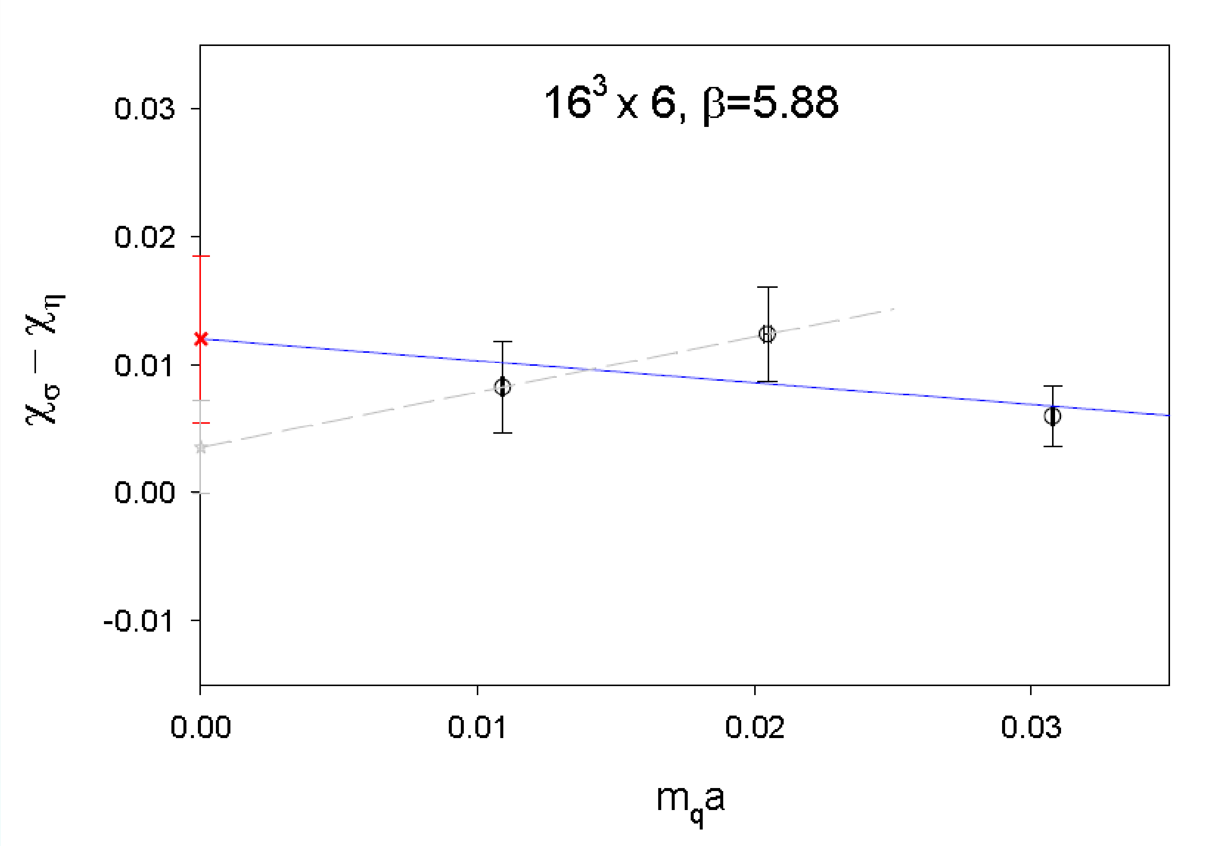
$T \approx 180$ MeV



$$\chi_\eta = \chi_\delta \Rightarrow \text{ restoration of } SU(2)_L \times SU(2)_R$$

Chiral Susceptibilities (cont.)

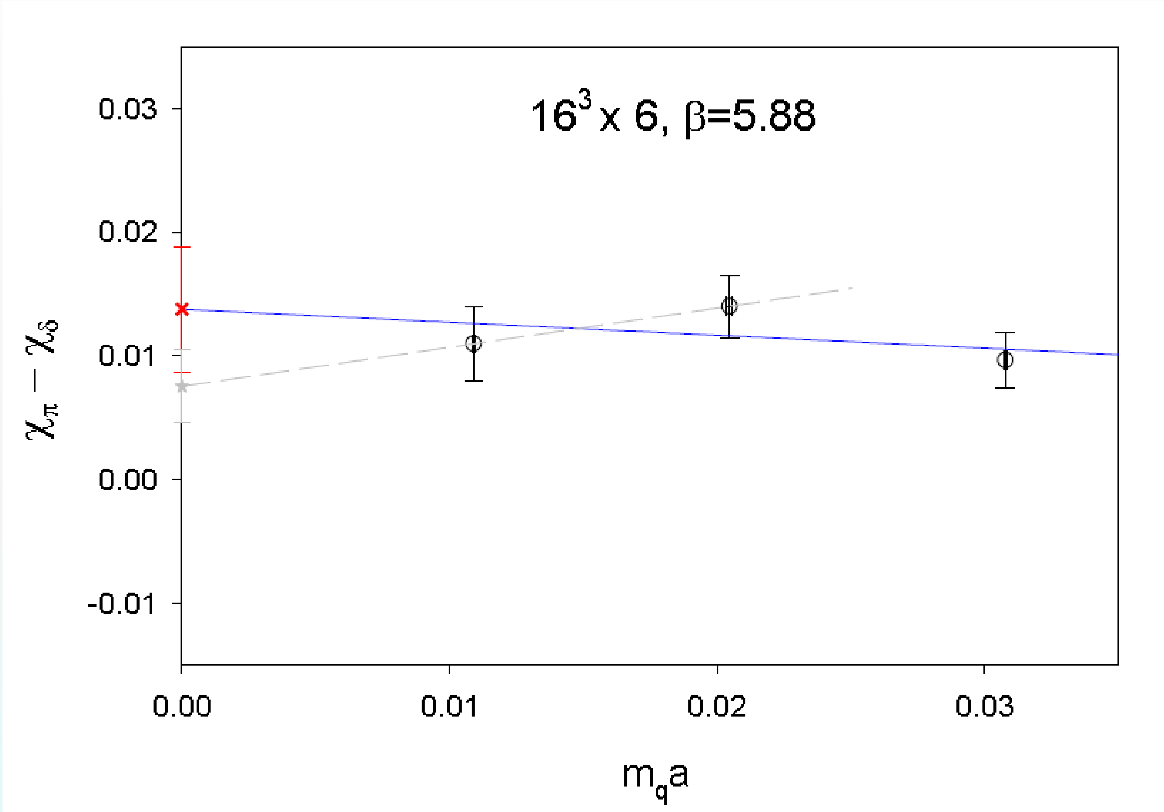
$T \approx 180$ MeV



If $\chi_\sigma = \chi_\eta$ in the chiral limit, then $U(1)_A$ is restored.

Chiral Susceptibilities (cont.)

$T \approx 180$ MeV

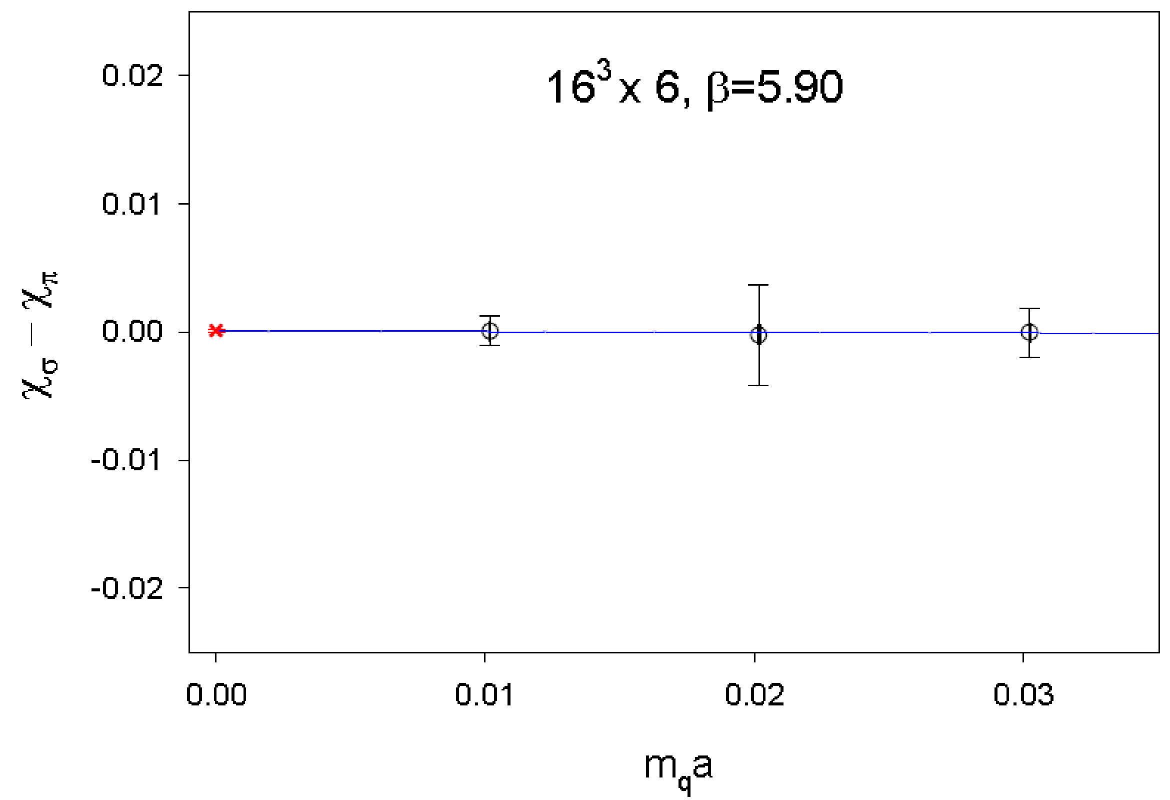


If $\chi_\pi = \chi_\delta$ in the chiral limit, then $U(1)_A$ is restored.

Chiral Susceptibilities (cont.)

$$\chi_\pi \simeq \chi_\sigma \simeq 0.235$$

$T \approx 210$ MeV

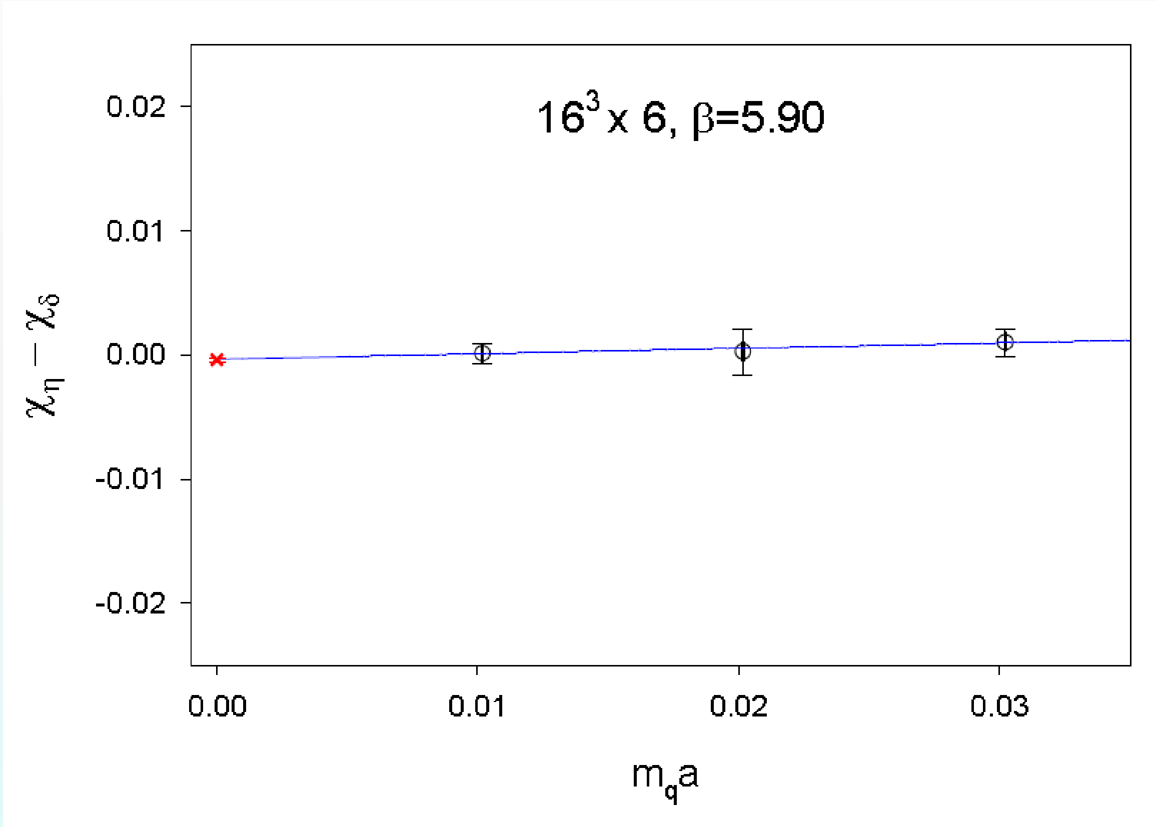


$$\chi_\pi = \chi_\sigma \Rightarrow \text{restoration of } SU(2)_L \times SU(2)_R$$

Chiral Susceptibilities (cont.)

$$\chi_\eta \simeq \chi_\delta \simeq 0.235$$

$T \approx 210$ MeV

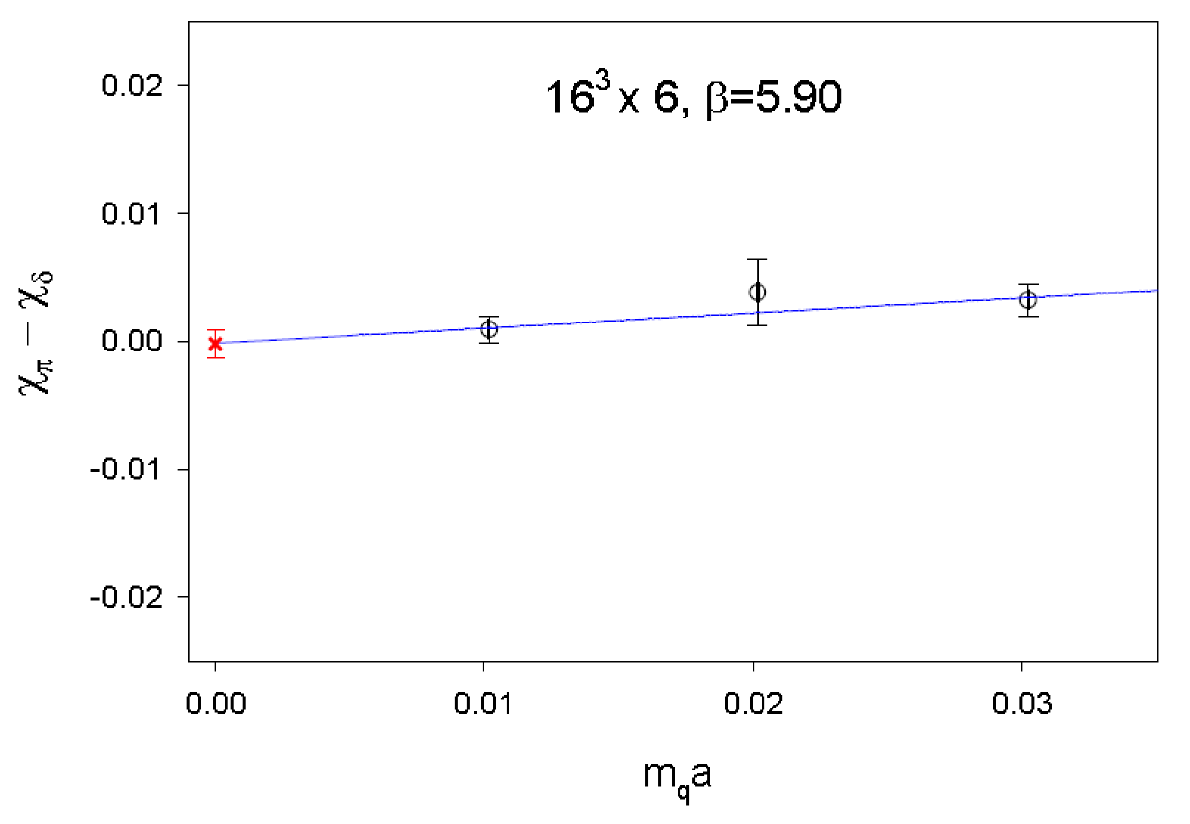


$\chi_\eta = \chi_\delta \Rightarrow$ restoration of $SU(2)_L \times SU(2)_R$

Chiral Susceptibilities (cont.)

$$\chi_\pi \simeq \chi_\delta \simeq 0.235$$

$T \approx 210$ MeV

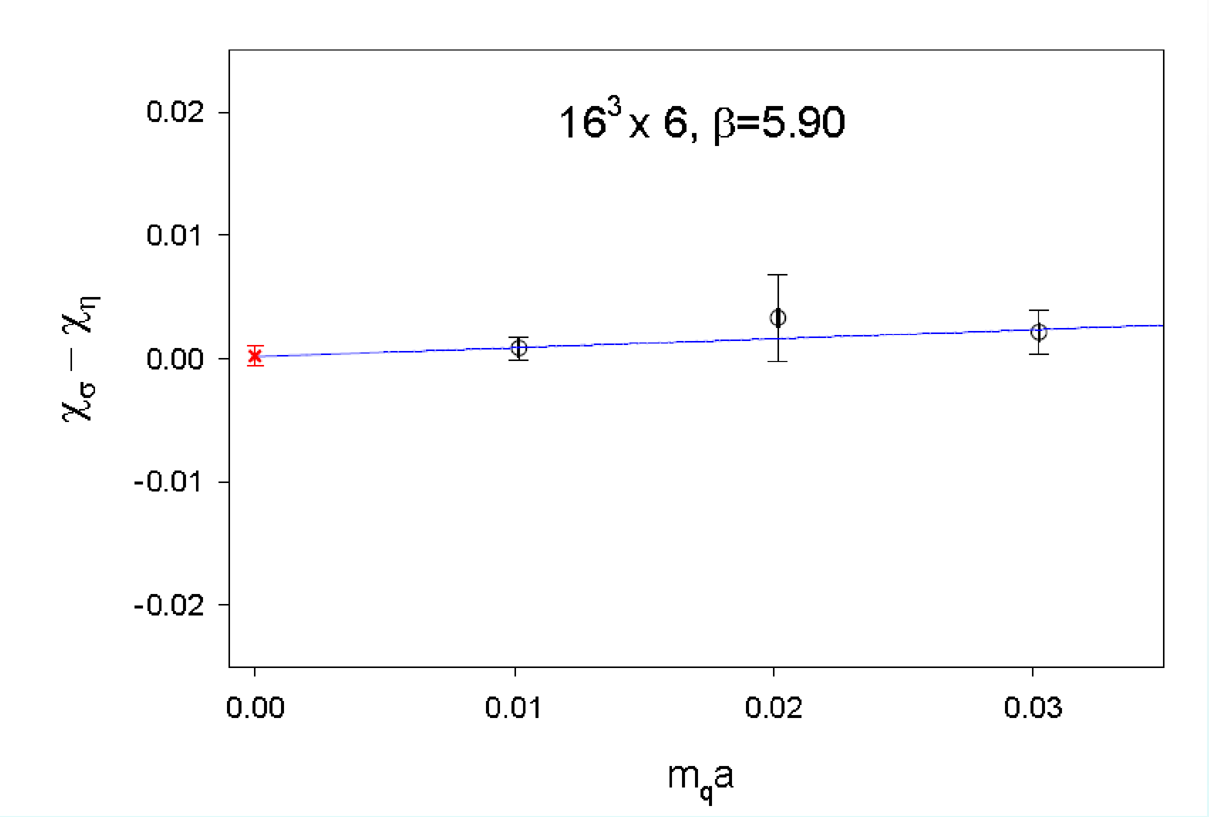


$$\chi_\pi = \chi_\delta \;\Rightarrow\; \text{restoration of } U(1)_A$$

Chiral Susceptibilities (cont.)

$$\chi_\eta \simeq \chi_\sigma \simeq 0.235$$

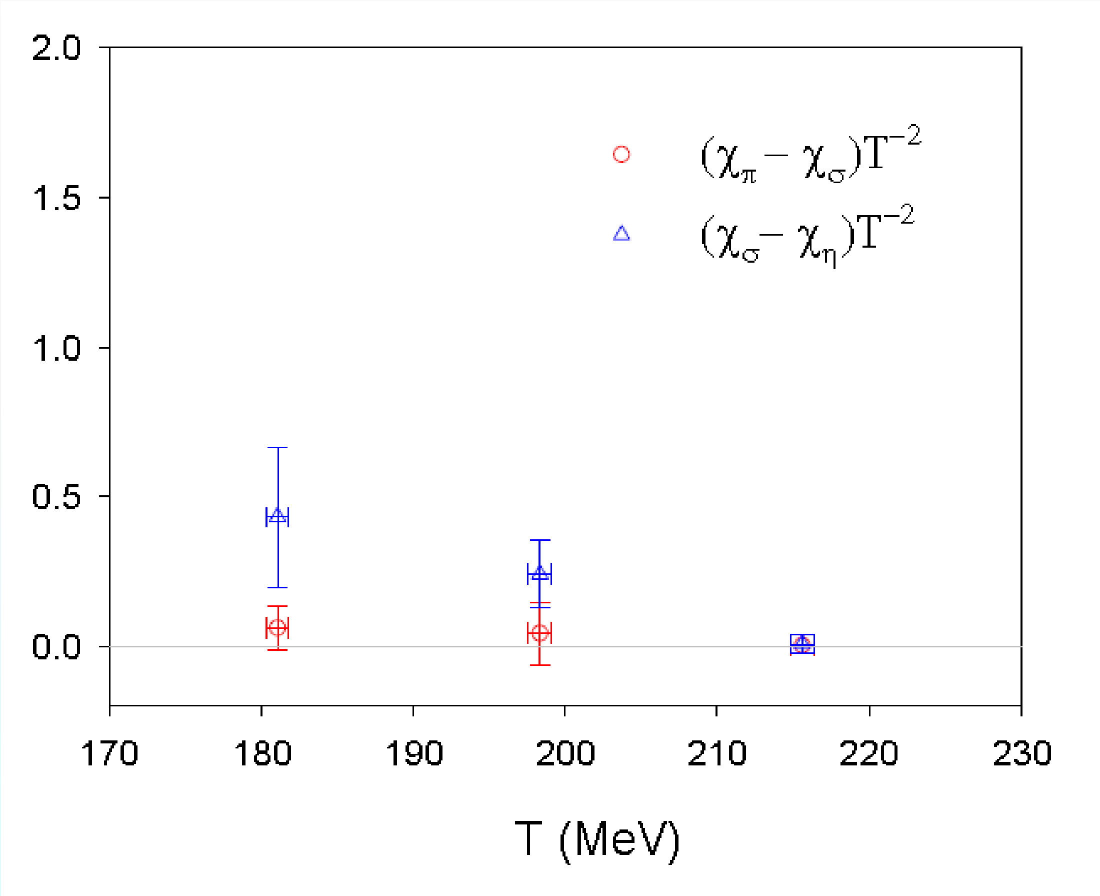
$T \approx 210$ MeV



$$\chi_\sigma = \chi_\eta \;\Rightarrow\; \text{restoration of } U(1)_A$$

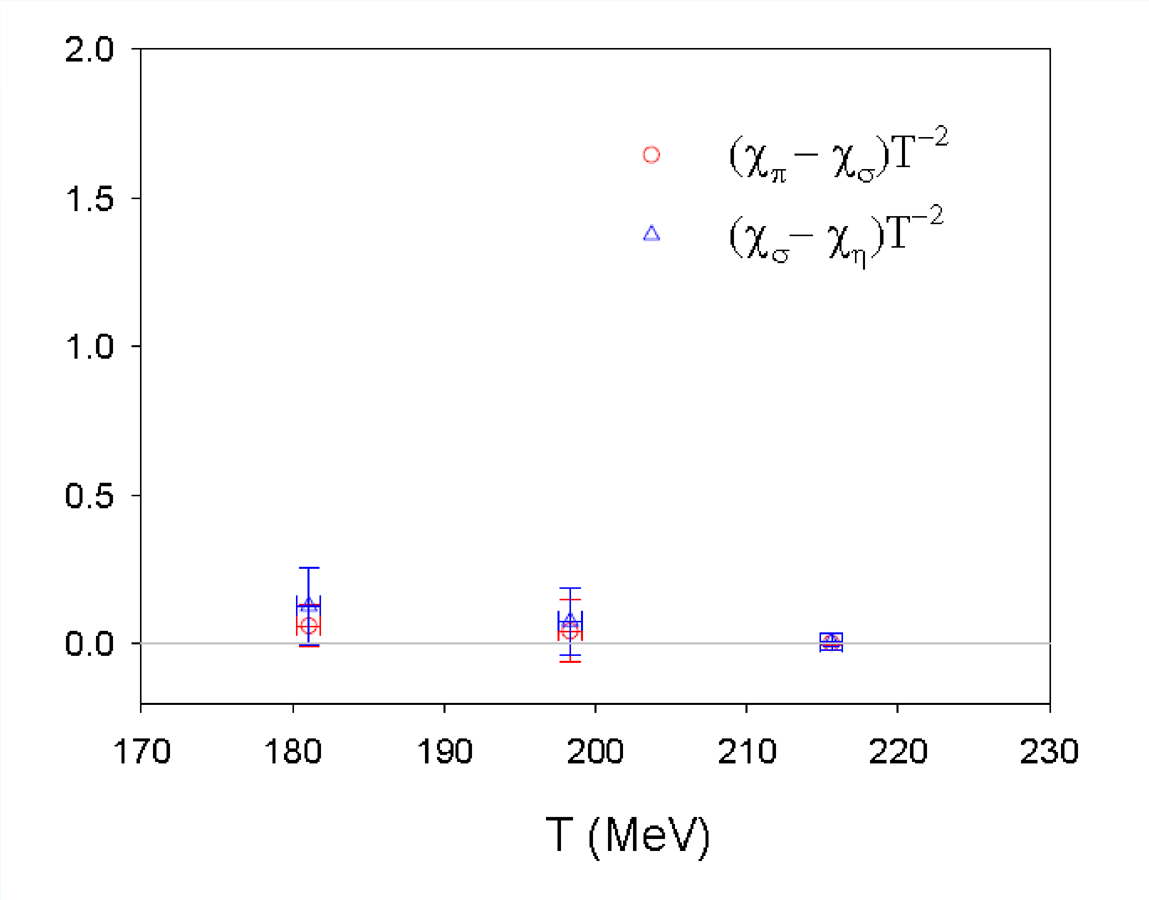
Chiral Susceptibilities (cont.)

Data points are obtained by chiral extrapolation with 3 quark masses, $m_q a = 0.01,\ 0.02, 0.03$



Chiral Susceptibilities (cont.)

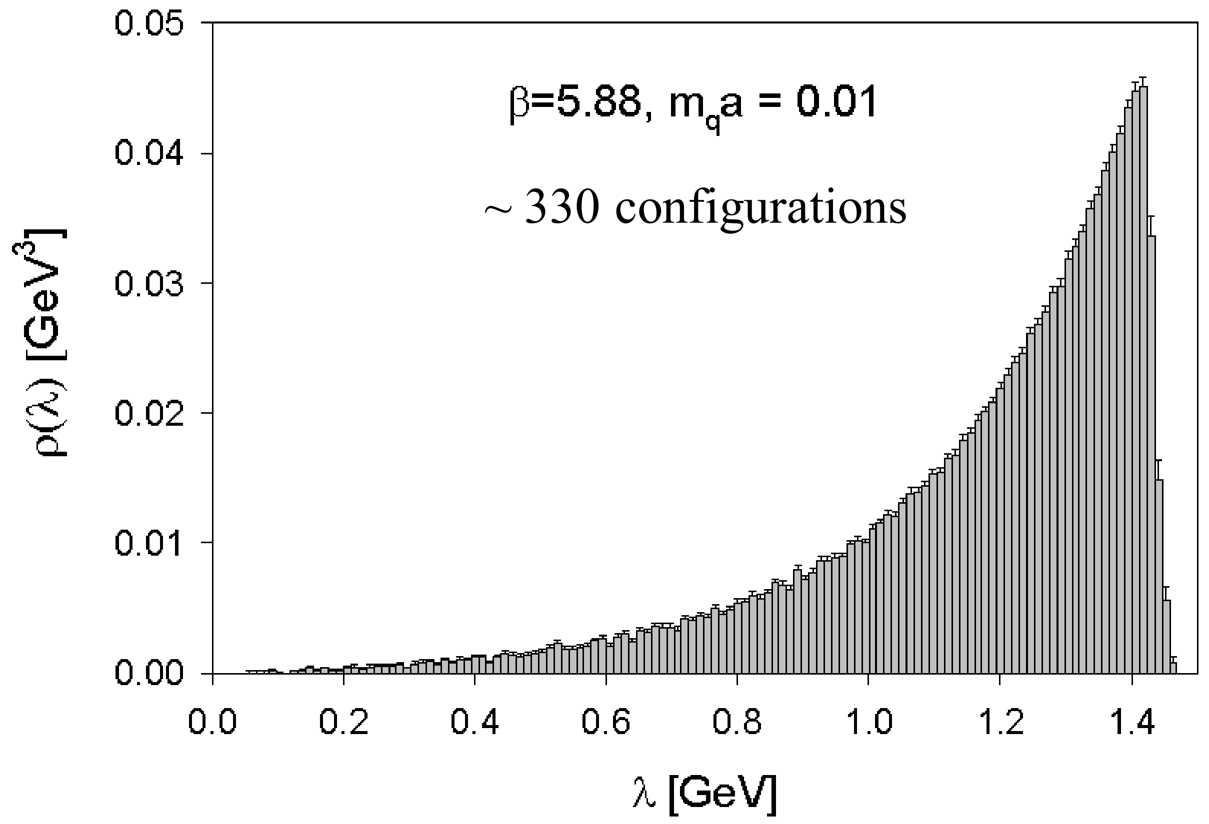
Data points are obtained by chiral extrapolation with 2 quark masses, $m_q a = 0.01,\ 0.02$



Eigenvalue density of the overlap operator

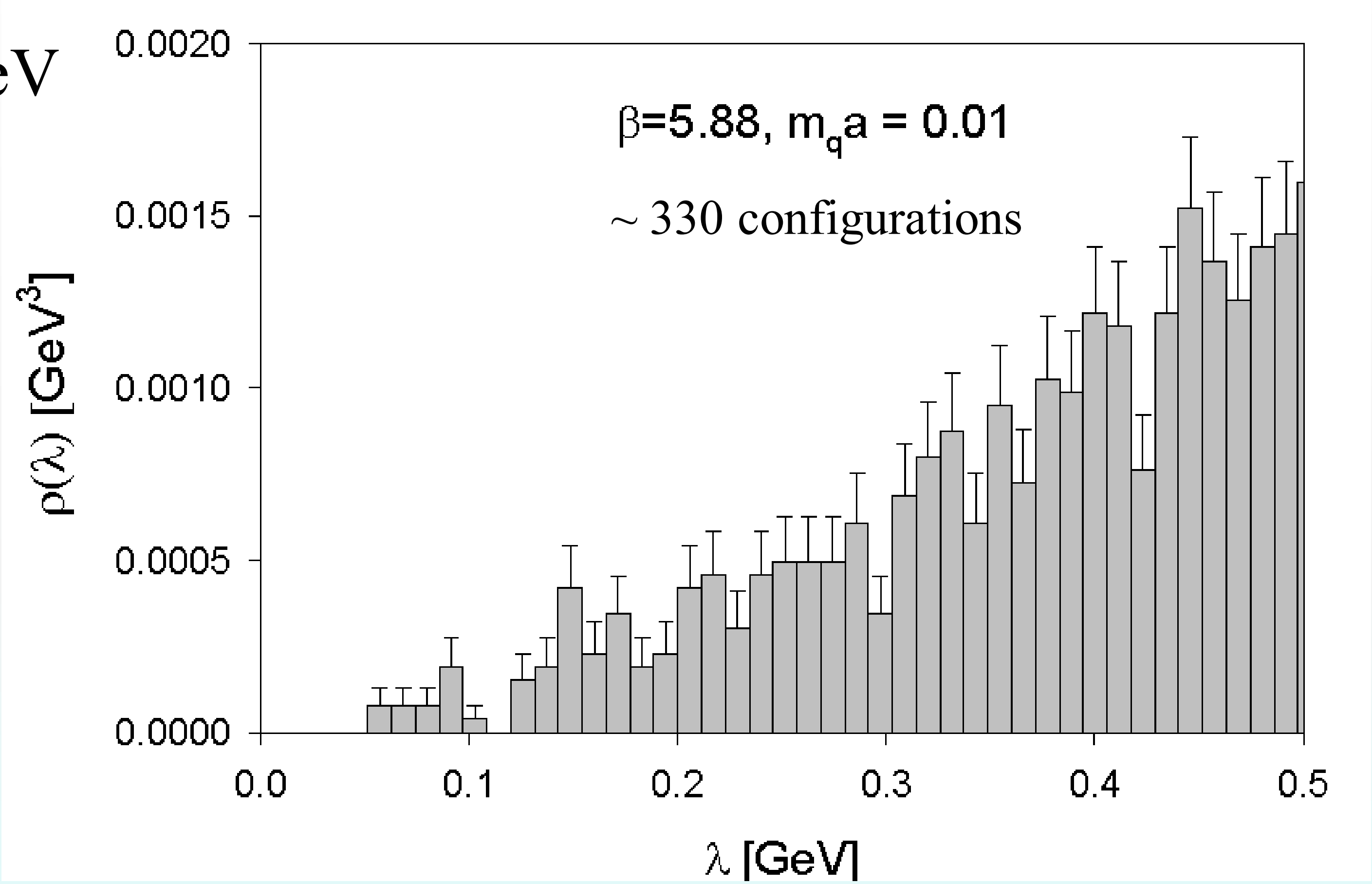
- For each conf, **zero modes** plus **200+200 conjugate pairs of low-lying eigenmodes** of the overlap operator are projected.

$T \approx 180$ MeV



Eigenvalue density of the overlap op. (cont)

$T \approx 180$ MeV

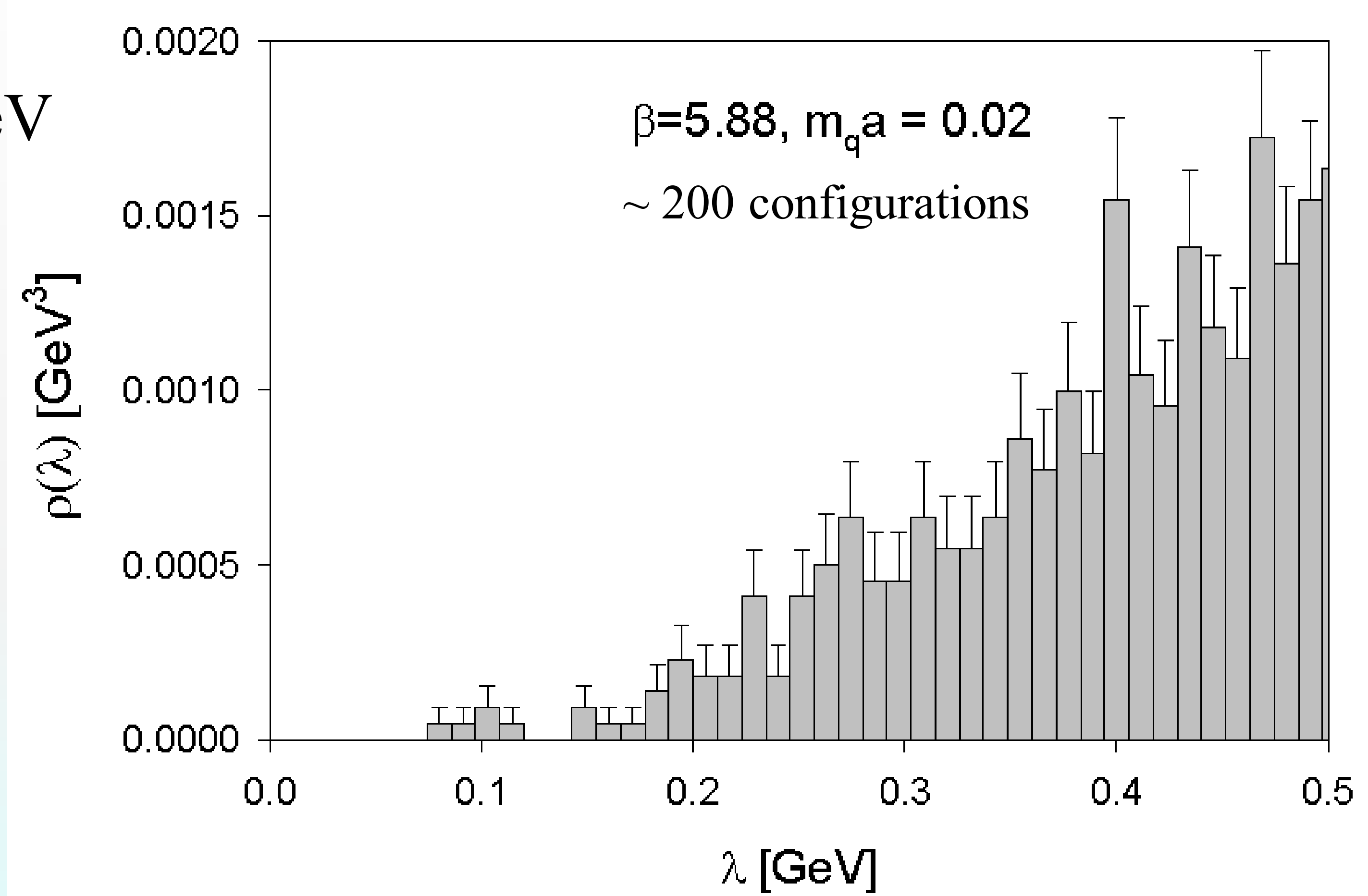


suppresion of low-modes stronger than

Aoki-Fukaya-Taniguchi [PRD, 2012], $\rho(\lambda) \approx \lambda^{\alpha},\ \alpha \geq 3$

Eigenvalue density of the overlap op. (cont)

$T \approx 180$ MeV

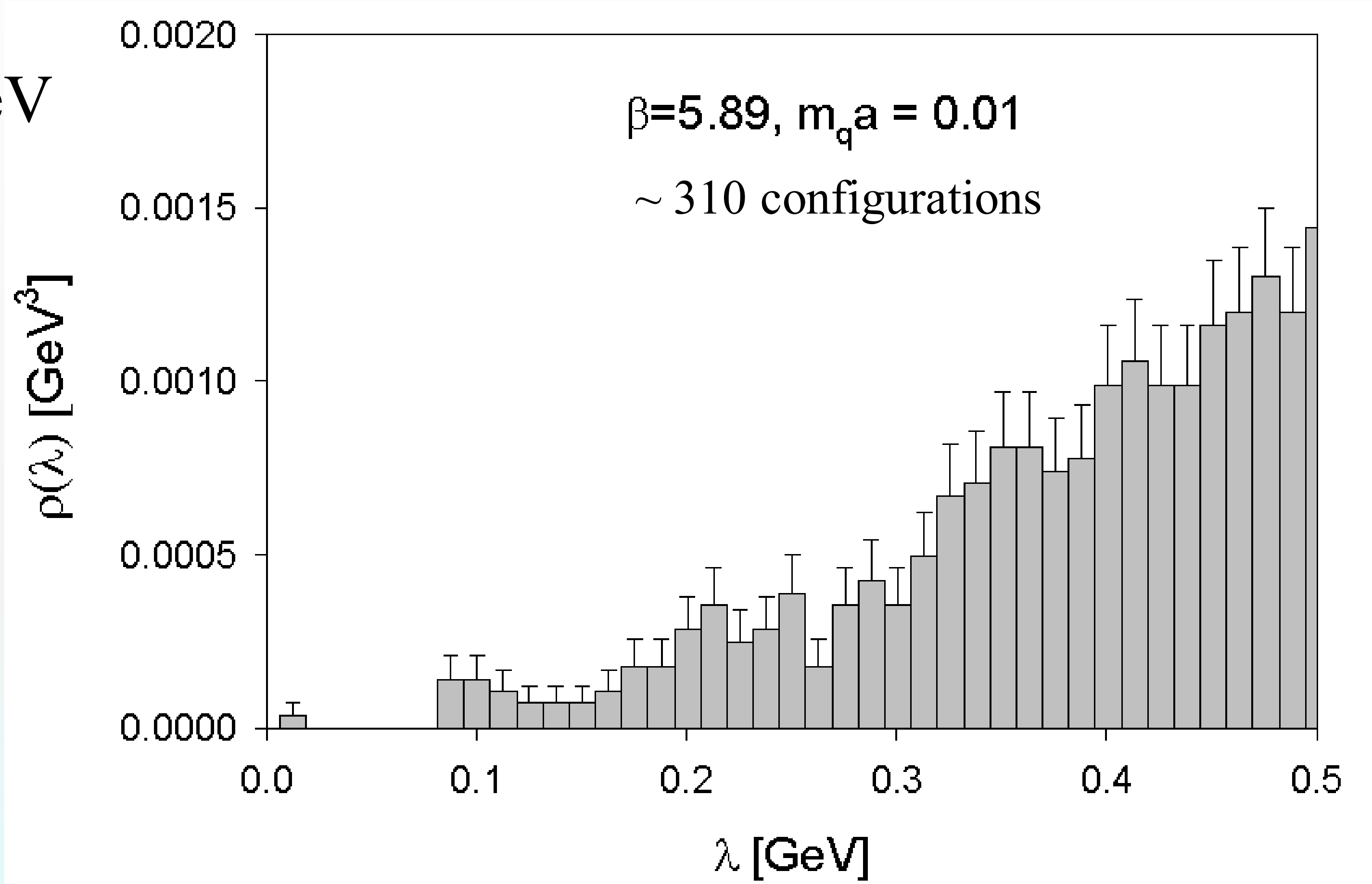


suppresion of low-modes stronger than

Aoki-Fukaya-Taniguchi [PRD, 2012], $\rho(\lambda) \approx \lambda^{\alpha}, \ \alpha \geq 3$

Eigenvalue density of the overlap op. (cont)

$T \approx 200$ MeV

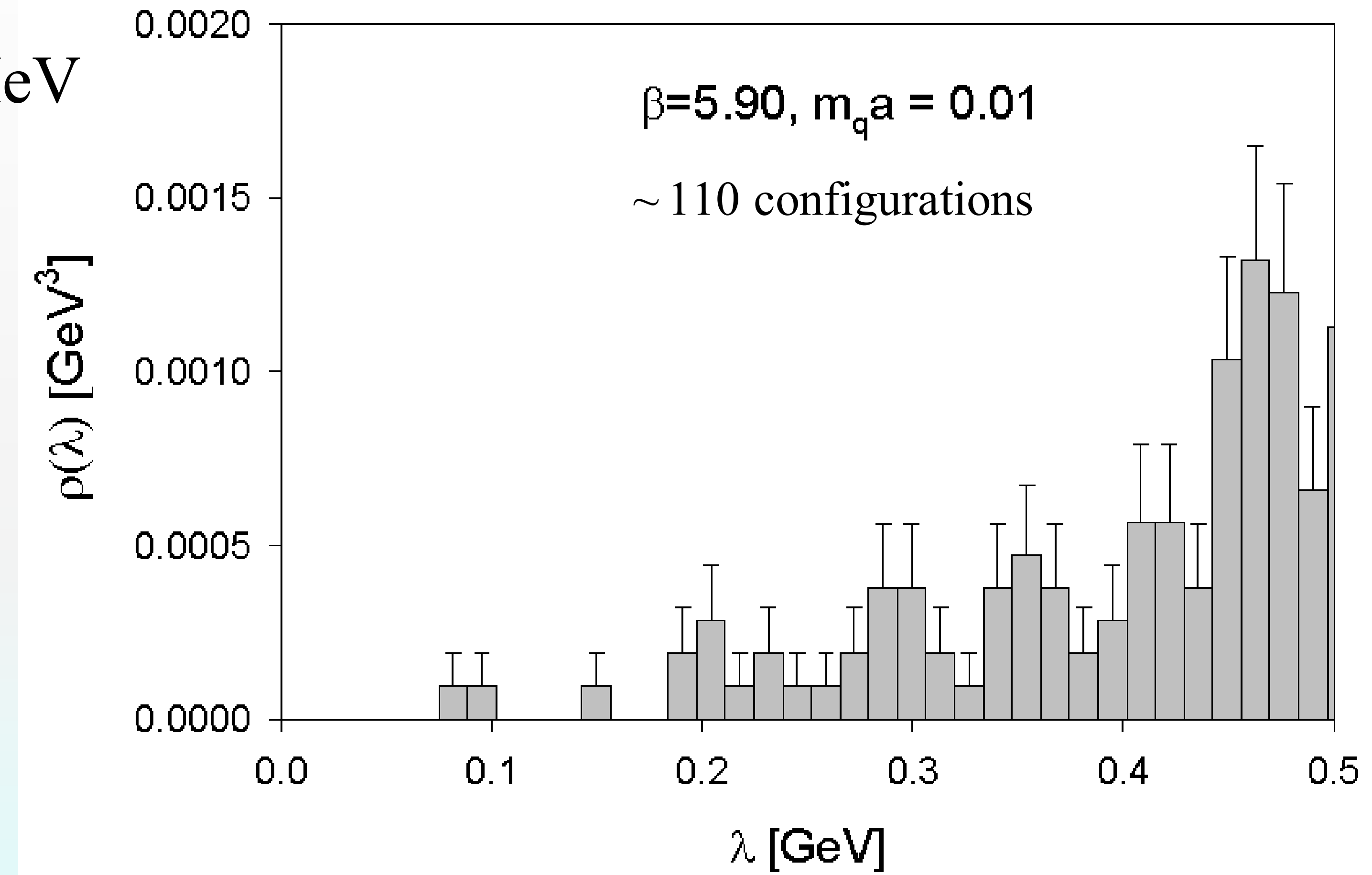


suppresion of low-modes stronger than

Aoki-Fukaya-Taniguchi [PRD, 2012], $\rho(\lambda) \approx \lambda^{\alpha}, \ \alpha \geq 3$

Eigenvalue density of the overlap op. (cont)

$T \approx 210$ MeV



suppresion of low-modes stronger than

Aoki-Fukaya-Taniguchi [PRD, 2012], $\rho(\lambda) \approx \lambda^{\alpha}, \ \alpha \geq 3$

Concluding Remarks

- TWQCD preliminary results of chiral susceptibilities and the eigenvalue density of the overlap operator suggest that $U(1)_A$ symmetry is likely to be restored at $T_1 \approx T_c$

- This implies that the chiral phase transition of 2-flavors QCD in the chiral limit is likely to be first order [Pisarski-Wilzcek, 1984], or second order in the $U(2)_L \times U(2)_R / U(1)_V$ universality class.

- A more precise determination of T_c and T_1 , with a finer scan in β, and also with a larger volume, are necessary to clarify these issues.