Fermion Bag Solutions
to Sign Problems
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in collaboration with Anyi Li
The QCD Sign Problem
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The QCD partition function

\[ Z = \int [dU] \ e^{-S_G(U)} \ \int d\bar{\psi} \ d\psi \ e^{-\bar{\psi}_x M_{xy} [U, U^\dagger]} \ \psi_y \]
The QCD Sign Problem

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Solution to sign problems in \( d > 2 \) usually imply

\[ M[U, U^\dagger] = \begin{pmatrix} 0 & D[U, U^\dagger] \\ -(D[U, U^\dagger])^\dagger & 0 \end{pmatrix} \]
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In the presence of a chemical potential

\[
M[U, U^\dagger, \mu] = \begin{pmatrix}
0 & D[Ue^\mu, U^\dagger e^{-\mu}] \\
-(D[Ue^{-\mu}, U^\dagger e^\mu])^\dagger & 0
\end{pmatrix}
\]
In the presence of a chemical potential

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0 & D[Ue^\mu, U^\dagger e^{-\mu}] \\
-(D[Ue^{-\mu}, U^\dagger e^\mu])^\dagger & 0
\end{pmatrix}
\]

Then,

\[
\text{Det}(M[U, U^\dagger, \mu]) = \text{Det}(D[Ue^\mu, U^\dagger e^{-\mu}]) \text{Det}(D^\dagger[Ue^{-\mu}, U^\dagger e^\mu])
\]

is no longer guaranteed to be positive!
In the presence of a chemical potential

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Then,

\[ \text{Det}(M[U, U^\dagger, \mu]) = \text{Det}(D[Ue^\mu, U^\dagger e^{-\mu}]) \text{Det}(D^\dagger[Ue^{-\mu}, U^\dagger e^\mu]) \]

is no longer guaranteed to be positive!

Loss of “pairing” is the origin of the sign problem
Sign Problems in Yukawa Models
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Consider the partition function

\[ Z = \int [dU] \ e^{-S_G(U)} \ \int [d\phi] \ e^{-S_b(\phi)} \ \int d\bar{\psi} \ d\psi \ e^{-\bar{\psi}_x M_{xy} [U, U^\dagger]} \ \psi_x - g \phi_x \bar{\psi}_x \psi_x \]
Sign Problems in Yukawa Models

Consider the partition function

$$Z = \int [dU] \ e^{-S_G(U)} \ \int [d\phi] \ e^{-S_b(\phi)} \ \int d\bar{\psi} \ d\psi \ e^{-\bar{\psi} M_{xy}[U,U^\dagger]} \ \psi_y - g_{\phi x} \bar{\psi}_x \psi_x$$

The fermion matrix is now given by

$$(\phi + M[U, U^\dagger]) = \begin{pmatrix} g_{\phi e} & D[U, U^\dagger] \\ - (D[U, U^\dagger])^\dagger & g_{\phi o} \end{pmatrix}$$

where $\phi_e$, $\phi_o$ are diagonal complex matrices.
Again, $\text{Det}(\phi + M[U, U^\dagger])$ is not guaranteed to be positive.
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Again, $\text{Det}(\phi + M[U, U^\dagger])$ is not guaranteed to be positive.

The Yukawa coupling can also destroy the "pairing" mechanism.
Take Home Lessons
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Chemical potential is not the only source of sign problems!
Take Home Lessons

Chemical potential is not the only source of sign problems!

Any interaction that destroys “pairing” can in principle lead to sign problems!
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Any interaction that destroys “pairing” can in principle lead to sign problems!

A new class of “Yukawa” sign problems are now solvable using the “fermion bag approach”.

Wednesday, August 7, 2013
The Fermion Bag Idea

SC, 2010
Group Fermion Worldlines
(fermion bags)
and sum over each group individually.
The Fermion Bag Idea
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(Extension of the meron cluster idea)
SC, Wiese, 2000
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Group Fermion Worldlines (fermion bags) and sum over each group individually.

(Extension of the meron cluster idea)

Choose fermion bags carefully that help solve sign problems

SC, 2010

SC, Wiese, 2000
Consider

\[ \int [d\bar{\psi} d\psi] \ e^{-\bar{\psi} M \psi} (-\bar{\psi}_i \psi_i)(-\bar{\psi}_j \psi_j)(-\bar{\psi}_k \psi_k)(-\bar{\psi}_l \psi_l)(-\bar{\psi}_m \psi_m) \]
Consider

\[ \int [d\bar{\psi} d\psi] \, e^{-\bar{\psi} \gamma^i M_{ij} \psi} (\bar{\psi}_{i_1} \psi_{i_1}) (\bar{\psi}_{i_2} \psi_{i_3} \bar{\psi}_{i_3} \psi_{i_2}) \]

\[ (\bar{\psi}_{i_2} \psi_{i_4} \bar{\psi}_{i_4} \psi_{i_2}) (\bar{\psi}_{i_3} \psi_{i_5} \bar{\psi}_{i_5} \psi_{i_3}) \]

\[ (\bar{\psi}_{i_6} \psi_{i_7} \bar{\psi}_{i_7} \psi_{i_6}) (\bar{\psi}_{i_8} \psi_{i_9} \bar{\psi}_{i_9} \psi_{i_8}) \]
Consider

\[
\int [d\bar{\psi} d\psi] \ e^{-\bar{\psi} M \psi} \ (-\bar{\psi}_{i_1} \psi_{i_1})(-\bar{\psi}_{i_2} \psi_{i_2})(-\bar{\psi}_{i_3} \psi_{i_3})(-\bar{\psi}_{i_4} \psi_{i_4})(-\bar{\psi}_{i_5} \psi_{i_5})(-\bar{\psi}_{i_6} \psi_{i_6})(-\bar{\psi}_{i_7} \psi_{i_7})(-\bar{\psi}_{i_8} \psi_{i_8})(-\bar{\psi}_{i_9} \psi_{i_9})
\]

\[= \ Det(W)\]
Consider
\[ \int [d\bar{\psi}d\psi] \, e^{-\bar{\psi}iMij\psi} \left( -\bar{\psi}_i \psi_i \right) \left( -\bar{\psi}_i \psi_i \right) \left( -\bar{\psi}_i \psi_i \right) \left( -\bar{\psi}_i \psi_i \right) \]
\[ \left( -\bar{\psi}_i \psi_i \right) \left( -\bar{\psi}_i \psi_i \right) \left( -\bar{\psi}_i \psi_i \right) \left( -\bar{\psi}_i \psi_i \right) \]
\[ \left( -\bar{\psi}_i \psi_i \right) \left( -\bar{\psi}_i \psi_i \right) \left( -\bar{\psi}_i \psi_i \right) \left( -\bar{\psi}_i \psi_i \right) \]
\[ = \text{Det}(W) \]

\( W \) is the matrix obtained by dropping some rows and the same columns from \( M \).
$$M = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0 \\
-M_{11}^* & -M_{21}^* & -M_{31}^* & \ldots & -M_{N1}^* \\
-M_{12}^* & -M_{22}^* & -M_{32}^* & \ldots & -M_{N2}^* \\
-M_{13}^* & -M_{23}^* & -M_{33}^* & \ldots & -M_{N3}^* \\
-M_{1N}^* & -M_{2N}^* & -M_{3N}^* & \ldots & -M_{NN}^* \\
\end{pmatrix}
\begin{pmatrix}
M_{11} & M_{12} & M_{13} & \ldots & M_{1N} \\
M_{21} & M_{22} & M_{23} & \ldots & M_{2N} \\
M_{31} & M_{32} & M_{33} & \ldots & M_{3N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
M_{N1} & M_{N2} & M_{N3} & \ldots & M_{NN} \\
\end{pmatrix}$$
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0 & 0 & 0 & \ldots & 0 \\
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\vdots & \vdots & \vdots & \ddots & \vdots \\
-M_{1N}^* & -M_{2N}^* & -M_{3N}^* & \ldots & -M_{NN}^* \\
\end{pmatrix}
\]

\[
W = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
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\end{pmatrix}
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\end{pmatrix}$$

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\end{pmatrix}$$
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\[ \text{Det}(W) \geq 0 \]
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A new class of “solvable” problems
A new class of “solvable” problems

Consider actions of the form

\[ S = \sum_{xy} \overline{\psi}_x M_{xy}[\sigma] \psi_x + g \sum_x \phi_x \overline{\psi}_x \psi_x + S_b(\sigma, \phi) \]
A new class of “solvable” problems

Consider actions of the form

\[ S = \sum_{xy} \bar{\psi}_x M_{xy}[\sigma] \psi_x + g \sum_x \phi_x \bar{\psi}_x \psi_x + S_b(\sigma, \phi) \]

The action \( S_b[\sigma, \phi] \) is chosen such that

the sign problem in the k-pt correlation function

\[ G(z_1, \ldots, z_k, \sigma) = \int [d\phi] e^{-S_b(\sigma, \phi)} \phi_{z_1} \phi_{z_2} \cdots \phi_{z_k} \]

is solvable.
Solvable bosonic theories are those in which we can write

\[ G(z_1, \ldots, z_k, \sigma) = \sum_b \int [d\rho] \, \Omega(\sigma, b, \rho, n), \]

\[ \Omega(\sigma, b, \rho, n) \geq 0 \]
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where the \([n]\) is a monomer field labeling the location of \(z_1, z_2, \ldots, z_k\).
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and \((b, \rho)\) are “other” bosonic fields introduced to solve the sign problem.
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and \((b, \rho)\) are “other” bosonic fields introduced to solve the sign problem.
These class of models are not solvable with the traditional approach
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\[ S = \bar{\psi}(M[\sigma] + g\Phi)\psi + S_b(\sigma, \phi) \]

\[ M[\sigma] + g\Phi = \begin{pmatrix} g \phi_1 & D[\sigma] \\ -D^\dagger[\sigma] & g \phi_2^* \end{pmatrix} \]

\[ Z = \int [d\sigma \ d\phi] e^{-S_b[\sigma, \phi]} \ Det(M[\sigma] + g\Phi) \]
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suffers from sign problem
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The Fermion bag approach solves the sign problem!
Fermion Bag approach
Fermion Bag approach

Rewrite the partition function as

\[ Z = \int [d\sigma \ d\phi] \ e^{-S_b(\sigma, \phi)} \ \int [d\bar{\psi}d\psi] \ e^{-\bar{\psi} \ M[\sigma] \ \psi} \ \prod_x \left( e^{-g \ \phi_x \ \bar{\psi}_x \ \psi_x} \right) \]
Fermion Bag approach

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\]

Due to the Grassmann nature

\[
e^{-g \ \phi_x \ \overline{\psi}_x \ \psi_x} = 1 + g \ \phi_x \ (-\overline{\psi}_x \ \psi_x) = \sum_{n_x = 0,1} \left( g \ \phi_x \ (-\overline{\psi}_x \ \psi_x) \right)^{n_x}
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Fermion Bag approach

Rewrite the partition function as

\[ Z = \int [d\sigma \ d\phi] \ e^{-S_b(\sigma, \phi)} \ \int [d\bar{\psi} d\psi] \ e^{-\bar{\psi} \ M[\sigma] \ \psi} \ \prod_x \ (e^{-g \ \phi_x \ \bar{\psi}_x \ \psi_x}) \]

Due to the Grassmann nature

\[ e^{-g \ \phi_x \ \bar{\psi}_x \ \psi_x} = 1 + g \ \phi_x (-\bar{\psi}_x \ \psi_x) = \sum_{n_x = 0,1} \ (g \ \phi_x (-\bar{\psi}_x \ \psi_x))^{n_x} \]

We can then rewrite

\[ Z = \sum [n] \ \int [d\sigma] \ \int [d\phi] \ e^{-S_b(\sigma, \phi)} \ \int [d\bar{\psi} d\psi] \ e^{-\bar{\psi} \ M[\sigma] \ \psi} \ \prod_x \ (g \ \phi_x (-\bar{\psi}_x \ \psi_x))^{n_x} \]
Consider a configuration \([n]\) where \(z_1\ z_2\ \ldots\ z_k\) are the \(k\) sites where \(n_x = 1\) and all other sites have \(n_x = 0\).
Consider a configuration \([n]\) where \(z_1 z_2 \ldots z_k\) are the \(k\) sites where \(n_x = 1\) and all other sites have \(n_x = 0\).
Consider a configuration \([n]\) where \(z_1, z_2, \ldots, z_k\) are the \(k\) sites where \(n_x = 1\) and all other sites have \(n_x = 0\).

\[
Z = \sum_{[n]} g^k \int [d\sigma] \int [d\phi] \ e^{-S_b(\sigma, \phi)} \ \phi_{z_1} \ \phi_{z_2} \ \cdots \ \phi_{z_k}
\]

\[
\int [d\bar{\psi} d\psi] \ e^{-\bar{\psi} \ M[\sigma] \ \psi} \ (-\bar{\psi}_{z_1} \psi_{z_1}) \ (-\bar{\psi}_{z_2} \psi_{z_2}) \ \cdots \ (-\bar{\psi}_{z_k} \psi_{z_k})
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Z = \sum_{[n]} g^k \int [d\sigma] \int [d\phi] \ e^{-S_b(\sigma,\phi)} \ \phi_{z_1} \ \phi_{z_2} \ \ldots \ \phi_{z_k}
\]

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\int [d\bar{\psi}d\psi] \ e^{-\bar{\psi} M[\sigma] \ \psi} \ (-\bar{\psi}_{z_1} \psi_{z_1}) \ (-\bar{\psi}_{z_2} \psi_{z_2}) \ \ldots \ (-\bar{\psi}_{z_k} \psi_{z_k})
\]
Consider a configuration $[n]$ where $z_1, z_2, \ldots, z_k$ are the $k$ sites where $n_x = 1$ and all other sites have $n_x = 0$.

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Z = \sum_{[n]} g^k \int [d\sigma] \int [d\phi] e^{-S_b(\sigma, \phi)} \phi_{z_1} \phi_{z_2} \ldots \phi_{z_k}
\]

\[
\int [d\bar{\psi} d\psi] e^{-\bar{\psi} M[\sigma] \psi} (\bar{\psi}_{z_1} \psi_{z_1}) (\bar{\psi}_{z_2} \psi_{z_2}) \ldots (\bar{\psi}_{z_k} \psi_{z_k})
\]

$G(z_1, \ldots, z_k, \sigma)$
Fermion correlation function

\[ \int [d\bar{\psi} d\psi] \ e^{-\bar{\psi} M[\sigma] \psi} \ \bar{\psi}_{z_1} \psi_{z_1} \cdots \bar{\psi}_{z_k} \psi_{z_k} \]
Fermion correlation function

\[ \int [d\bar{\psi}d\psi] \ e^{-\bar{\psi} M[\sigma] \psi} \ \bar{\psi}_{z_1} \psi_{z_1} \ ... \ \bar{\psi}_{z_k} \psi_{z_k} \]
Fermion correlation function

\[
\int [d\bar{\psi} d\psi] \ e^{-\bar{\psi} M[\sigma] \psi} \ \bar{\psi}_{z_1} \psi_{z_1} \ldots \bar{\psi}_{z_k} \psi_{z_k} = \text{Det}(W[n, \sigma]) \geq 0
\]
Fermion correlation function

\[
\int [d\bar{\psi} d\psi] \ e^{-\bar{\psi} \ M[\sigma] \ \psi} \ \bar{\psi}_{z_1} \ \psi_{z_1} \ \ldots \ \bar{\psi}_{z_k} \ \psi_{z_k} \\
= \ \text{Det}(W[n, \sigma]) \geq 0
\]

W is a (V-k) x (V-k) matrix obtained by dropping sites z_1 \ldots z_k in M
Fermion correlation function

\[
\int [d\bar{\psi} d\psi] \ e^{-\bar{\psi} M[\sigma] \psi \ \bar{\psi}_1 \psi_1 \ldots \bar{\psi}_k \psi_k}
\]

\[= \ \text{Det}(W[n, \sigma]) \geq 0\]

W is a \((V-k) \times (V-k)\) matrix obtained by dropping sites \(z_1 \ldots z_k\) in M

\[
M[\sigma] = \begin{pmatrix} 0 & D[\sigma] \\ -D^\dagger[\sigma] & 0 \end{pmatrix}
\]

\[
W[n, \sigma] = \begin{pmatrix} 0 & \tilde{D}[n, \sigma] \\ -\tilde{D}^\dagger[n, \sigma] & 0 \end{pmatrix}
\]
Fermion correlation function

\[
\int [d\bar{\psi} d\psi] e^{-\psi} M[\sigma] \psi \bar{\psi}_{z_1} \psi_{z_1} \ldots \bar{\psi}_{z_k} \psi_{z_k} = \text{Det} (W[n, \sigma]) \geq 0
\]

W is a (V-k) x (V-k) matrix obtained by dropping sites \(z_1 \ldots z_k\) in M

\[
M[\sigma] = \begin{pmatrix}
0 & D[\sigma] \\
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\end{pmatrix}
\]

\[
W[n, \sigma] = \begin{pmatrix}
0 & \tilde{D}[n, \sigma] \\
-\tilde{D}^\dagger[n, \sigma] & 0
\end{pmatrix}
\]

Connection to Diagrammatic Determinantal MC
Talks by Endres and Detmold
Thus, the partition function is given by

$$Z = \sum_{n,b} \int [d\sigma \ d\rho] \ g^k \ \Omega(\sigma, b, \rho, n) \ \text{Det}(W[n, \sigma])$$
Thus, the partition function is given by

\[ Z = \sum_{n,b} \int [d\sigma \ d\rho] \ g^k \ \Omega(\sigma, b, \rho, n) \ \text{Det}(W[n, \sigma]) \]
Thus, the partition function is given by

\[ Z = \sum_{n,b} \int [d\sigma \; d\rho] \; g^k \; \Omega(\sigma, b, \rho, n) \; \text{Det}(W[n, \sigma]) \]

No sign problem!
Thus, the partition function is given by

$$Z = \sum_{n,b} \int [d\sigma \, d\rho] \ g^k \ \Omega(\sigma, b, \rho, n) \ \text{Det}(W[n, \sigma])$$

No sign problem!

Interesting mapping into classical statistical mechanics
Another class of “solvable” problems
Another class of “solvable” problems

Consider actions of the form

\[ S = \sum_{xy} \overline{\psi}_x M_{xy}[\sigma] \psi_x - i \sum_x \left( g_1 \phi_1 x \psi_x^T \sigma_2 \psi_x - g_2 \phi_2 x \overline{\psi}_x \sigma_2 \overline{\psi}_x^T \right) + S_b(\sigma, \phi_1, \phi_2) \]

where \( \psi_x, \overline{\psi}_x \) are two component Grassmann fields
Another class of “solvable” problems

Consider actions of the form

\[ S = \sum_{xy} \bar{\psi}_x M_{xy} [\sigma] \psi_x - i \sum_x (g_1 \phi_1 x \psi_x^T \sigma_2 \psi_x - g_2 \phi_2 x \bar{\psi}_x \sigma_2 \bar{\psi}_x^T) + S_b (\sigma, \phi_1, \phi_2) \]

where \( \psi_x, \bar{\psi}_x \) are two component Grassmann fields

Assume \( M_{xy} = A_{xy} \otimes I + i B_{xy}^a \otimes \sigma^a \) with real \( A, B^a \)
Another class of “solvable” problems

Consider actions of the form

\[ S = \sum_{xy} \overline{\psi}_x M_{xy}[\sigma] \psi_x - i \sum_x \left( g_1 \phi_1 \psi_x^T \sigma_2 \psi_x - g_2 \phi_2 \overline{\psi}_x \sigma_2 \overline{\psi}_x^T \right) + S_b(\sigma, \phi_1, \phi_2) \]

where \( \psi_x, \overline{\psi}_x \) are two component Grassmann fields

Assume \( M_{xy} = A_{xy} \otimes I + i B_{xy}^a \otimes \sigma^a \) with real \( A, B^a \)

\[
M = \begin{pmatrix} C & D \\ -D^* & C^* \end{pmatrix} \quad W = \begin{pmatrix} \tilde{C} & \tilde{D} \\ -\tilde{D}^* & \tilde{C}^* \end{pmatrix}
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Such problems naturally describe “pairing” of fermions like in a superconductor
\[
M = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & \cdots & C_{1N} & M_{11} & M_{12} & M_{13} & \cdots & M_{1N} \\
C_{21} & C_{22} & C_{23} & \cdots & C_{2N} & M_{21} & M_{22} & M_{23} & \cdots & M_{2N} \\
C_{31} & C_{32} & C_{33} & \cdots & C_{3N} & M_{31} & M_{32} & M_{33} & \cdots & M_{3N} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
C_{N1} & C_{N2} & C_{N3} & \cdots & C_{NN} & M_{N1} & M_{N2} & M_{N3} & \cdots & M_{NN} \\
-M_{11}^* & -M_{12}^* & -M_{13}^* & \cdots & -M_{1N}^* & C_{11}^* & C_{12}^* & C_{13}^* & \cdots & C_{1N}^* \\
-M_{21}^* & -M_{22}^* & -M_{23}^* & \cdots & -M_{2N}^* & C_{21}^* & C_{22}^* & C_{23}^* & \cdots & C_{2N}^* \\
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\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-M_{1N}^* & -M_{2N}^* & -M_{3N}^* & \cdots & -M_{NN}^* & C_{N1}^* & C_{N2}^* & C_{N3}^* & \cdots & C_{NN}^* 
\end{pmatrix}
\]
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\end{pmatrix}
$$

$$W = \begin{pmatrix}
C_{11} & C_{12} & C_{13} & \ldots & C_{1N} & M_{11} & M_{12} & M_{13} & \ldots & M_{1N} \\
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\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
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solvable form
Other classes of solvable models
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Begin with any class of matrices with some property that gives positive determinants.
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Introduce interactions that delete rows and columns but preserve the property of positive determinants
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Such interactions can be coupled to bosonic “solvable” bosonic degrees of freedom
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Challenge: Understand “solvability” with non-Abelian fields

Dual Variables(?)
Subset methods, Bloch’s Talk
A QCD-like Polyakov-Loop Model may be “solvable” (?)
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Action

\[ S = \sum_{xy} \overline{\psi}_x M_{xy} [z, z^*, \mu] \psi_y + S_b(z) \]

- Massless staggered Dirac operator
- \( Z_3 \) Polyakov-Loop variables
- \( Z_3 \) Potts Model
A QCD-like Polyakov-Loop Model may be “solvable” (?)

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massless staggered Dirac operator

\[ Z_3 \] Potts Model

\[ Z_3 \] Polyakov-Loop variables

Constraint:
The Polyakov-Loop variables live only on alternate time slices!
Some repulsive models also solvable!
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\[ S = \sum_{xy} \overline{\psi}_x D_{xy} \psi_x + \overline{\chi}_x D_{xy} \chi_x \]

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attractive  repulsion
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Fermion Bag Configuration

fermion bag containing species 1

fermion bag containing species 2
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No sign problem!

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fermion bag containing species 1

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attractive

repulsion

Wednesday, August 7, 2013
MC Results: Four-Fermion Models

S.C. A.Li, PRL (2012), arXiv:1304.7761
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SU(2) × U(1) symmetric models

\[ S(\overline{\psi}, \psi) = \sum_{xy} \overline{\psi}_x M_{xy} \psi_y - \sum_{\langle xy \rangle} U_{\langle xy \rangle} \overline{\psi}_x \psi_x \overline{\psi}_y \psi_y \]
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Solvable with HMC

Thirring

massless fermions/ U(1) symmetric

Gross-Neveu

massive fermions/ U(1) broken

suffers from sign problems in HMC
Thirring model results

Combined fit results

\( U_c = 0.2608(2) \)
\( \nu = 0.85(1) \)
\( \eta = 0.65(1) \)
\( \eta_\psi = 0.37(1) \)
Gross-Neveu Model Results

$\chi / L^{2n}$ vs $U$

- $Z_2$
- $U(1)$

Wednesday, August 7, 2013
## Comparison: Old vs New

<table>
<thead>
<tr>
<th>Model</th>
<th>Symmetry</th>
<th>Work</th>
<th>( \nu )</th>
<th>( \eta )</th>
<th>( \eta_{\psi} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>N=1 Lattice-GN</td>
<td>SU(2) x Z(_2)</td>
<td>Karkkainen, et al. (1994)</td>
<td>1.00(4)</td>
<td>0.756(8)</td>
<td>-</td>
</tr>
<tr>
<td>N=1 Lattice GN</td>
<td>SU(2) x Z(_2)</td>
<td>SC &amp; Li (2012)</td>
<td>0.83(1)</td>
<td>0.62(1)</td>
<td>0.38(1)</td>
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<tr>
<td>N = 1 Lattice-Th</td>
<td>SU(2)x U(1)</td>
<td>Debbio, et al., (1997)</td>
<td>0.80(15)</td>
<td>0.70(15)</td>
<td>-</td>
</tr>
<tr>
<td>N = 1 Lattice-Th</td>
<td>SU(2)x U(1)</td>
<td>Barbour et. al., (1998)</td>
<td>0.80(20)</td>
<td>0.4(2)</td>
<td>-</td>
</tr>
<tr>
<td>N=1 Lattice-(GN/Th)</td>
<td>SU(2) x U(1)</td>
<td>SC &amp; Li (2013)</td>
<td>0.849(8)</td>
<td>0.633(8)</td>
<td>0.373(3)</td>
</tr>
</tbody>
</table>
Summary
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- Fermion-bags is a general idea which has already solved many new sign problems in Yukawa models that seemed unsolvable earlier.

- Precision Quantum Critical Behavior in a class of Fermi systems is within reach.