Dual Methods for Lattice Field Theories at Finite Density

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Sign Problem, Complex Action Problem

Non-zero chemical Potential $\mu$ in lattice QCD $\rightarrow$ $\det[D] \in \mathbb{C}$

$\rightarrow$ No phase diagram with standard Monte Carlo techniques!

Similar problems for other (lattice) field theories

Complex action $S$ for $\mu \neq 0$

$\rightarrow$ Boltzmann factor $e^{-S} \in \mathbb{C} \rightarrow$ no probabilistic interpretation

Model dependent solution: reformulation in terms of dual variables

Start with simple models and try to generalize!
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Dual Formulation – General Idea

**Partition function** $Z \in \mathbb{R}$

$$Z = \int D[\phi] \ e^{-S[\phi]} \quad \text{with} \quad e^{-S} \in \mathbb{C} \quad \text{and} \quad \phi \ldots \text{“conventional fields”}$$

→ try to find representation, such that

$$Z = \sum_{\{l\}} W[l] \quad \text{with} \quad W \in \mathbb{R}^+ \quad \text{and} \quad l \ldots \text{new degrees of freedom}$$

→ probabilistic interpretation

$$P[l] \equiv \frac{W[l]}{Z} \quad \ldots \quad \text{probability weight of configuration} \ l$$
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$\phi^4$ Model on the Lattice

Continuum action

$$S = \int d^4x \left[ |\partial_\nu \phi|^2 + (m^2 - \mu^2) |\phi|^2 + \lambda |\phi|^4 + \mu (\phi^* \partial_4 \phi - \partial_4 \phi^* \phi) \right]$$

Lattice action

$$S = \sum_x \left( -\sum_{\nu=1}^{4} \left( e^{\mu \delta_{\nu,4}} \phi_x \phi_{x+\hat{\nu}}^* + e^{-\mu \delta_{\nu,4}} \phi_x^* \phi_{x+\hat{\nu}} \right) + \kappa |\phi_x|^2 + \lambda |\phi_x|^4 \right)$$

with kinetic, mass and self-interaction terms.

Chemical potential $\mu$ couples only in 4-direction (time).

→ Complex action problem for $\mu \neq 0$. 
Dual Formulation of the $\phi^4$ Model – Sketch

Partition function

\[ Z = \int \mathcal{D}[\phi] \ e^{-S} \sim \int \mathcal{D}[\phi] \ \prod_{x,\nu} e^{S_{x,\nu}} \sim \int \mathcal{D}[\phi] \ \prod_{x,\nu} \sum_{l_{x,\nu}=0}^{\infty} \frac{(S_{x,\nu})^{l_{x,\nu}}}{l_{x,\nu}!} \]

Integrating out the original fields $\phi_x$, $\phi_x^*$ in terms of radial and angular parts ($\phi_x = r_x \ e^{i \theta_x}$)

→ Kronecker deltas (constrain the summation variables)

→ Weight factors (numerical integrals)
Partition function in dual representation

\[
Z = \sum_{\{k,m\}} \prod_{x,\nu} \frac{1}{(|k_{x,\nu}| + m_{x,\nu})!} \frac{1}{m_{x,\nu}!} \prod_{x} \delta \left( \sum_{\nu} [k_{x,\nu} - k_{x-\hat{\nu},\nu}] \right) \\
\times \prod_{x} e^{\mu k_{x,4}} \mathcal{W} \left( \sum_{\nu} [ |k_{x,\nu}| + |k_{x-\hat{\nu},\nu}| + 2(m_{x,\nu} + m_{x-\hat{\nu},\nu}) ] \right)
\]

- **New degrees of freedom** (dual variables, fluxes)

  \( k_{x,\nu} \in \mathbb{Z} \) (constrained) and \( m_{x,\nu} \in \mathbb{N}_0 \) (unconstrained)

- **Weight factors** \( \mathcal{W}(n_x) \) (numerical integrals)

- **Complex action problem is gone**

  \[
  Z = \sum_{\{k,m\}} P[k,m] \quad \text{with} \quad P[k,m] \in \mathbb{R}^+ \sim \text{probability weight}
  \]

- **Exact rewriting**
Numerical Simulation

\(\text{constraint} \iff \text{flux conservation} \) at each lattice site (closed loops)

\[\rightarrow\] Generalization of the Prokof’ev-Svistunov worm algorithm

- Standard Metropolis update sweep for the \(m\)-variables
- Worm update for the restricted \(k\)-variables
Results

Particle number density

\[ \langle n \rangle \propto \frac{\partial \ln Z}{\partial \mu} \propto \left\langle \sum_x k_{x,4} \right\rangle \sim \text{winding number} \]

The Silver Blaze Phenomenon

At \( T = 0 \) physics is independent of \( \mu < \mu_{\text{crit}} \).

second order phase transition at

\[ \mu_{\text{crit}} = m_{\text{ren}} = 1.146(1) \]

\[ \kappa = 9, \ \lambda = 1, \ T = 0 \]
Analysis at different temperatures (and observables, parameter sets, . . .)

→ Phase diagram and relativistic Bose condensation

Vertical sections below and above \( \mu_{\text{crit}} \).
2-Point Functions

Correlators

\[ \langle \phi_a \phi^*_b \rangle = \frac{1}{Z} \int \mathcal{D}[\phi] \; e^{-S} \; \phi_a \phi^*_b \equiv \frac{1}{Z} \; Z_{a,b} \]

cannot be expressed as partial derivatives of \( Z \).

Dual representation of \( Z_{a,b} \)

\[ \sum \prod \delta \left( \sum_{\nu} \ldots - \delta_{x,a} + \delta_{x,b} \right) \mathcal{W} \left( \sum_{\nu} \ldots + \delta_{x,a} + \delta_{x,b} \right) \left( \ldots \right) \]

\( \rightarrow \) modified arguments of

- Kronecker deltas and
- Weightfactors at lattice sites \( a \) and \( b \).
Numerical Simulation

New Constraint for $Z_{a,b}$

\[ Z \equiv \sum_{a,b} Z_{a,b} \sim \text{closed loops (} a = b \text{) } + \text{ open lines (} a \neq b \text{)} \]

Enlarged ensemble (Korzec et. al., Comput. Phys. Commun. 182 (2011))

Dual zero momentum temporal correlator

\[ C(t) \propto \langle \delta_{t, a_4 - b_4} \rangle_Z \propto e^{-mt} \]
\[ \mu_{\text{crit}} = 0.170(1) \]

\[ m_{\text{red}} \equiv E_{\pm} \pm \mu \]

\[ E_{+} = 0.1687(3) \]

\[ E_{-} = 0.1684(3) \]
Fits to the slopes behave exactly as expected!
Conclusion

- Different models can be mapped to a dual representation → **Complex action problem solved.**

- **Physical Observables** (e.g. particle number) and properties (e.g. phase diagram) can be studied **in terms of dual variables.**

- Generalization allows us to carry out **spectroscopy calculations** for **non-zero chemical potential.**

Outlook

- Continue work on correlators.

- Find dual formulation for models with non-abelian degrees of freedom.

- At the moment: SU(2) spin model

\[ Z = \int \mathcal{D}[U] \prod_{x, \nu} e^{\beta \text{Tr}[U_x U_{x+\nu}^\dagger]} + h \text{Tr}[U_x] \quad U_x \in SU(2) \]

- Successfully reformulated.