Dual Methods for Lattice Field Theories at Finite Density

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Sign Problem, Complex Action Problem

Non-zero chemical Potential μ in lattice QCD $\rightarrow$ $\det[D] \in \mathbb{C}$ ✗

$\rightarrow$ **No phase diagram with standard Monte Carlo techniques!**

Similar problems for other (lattice) field theories

Complex action S for $\mu \neq 0$ ✗

$\rightarrow$ Boltzmann factor $\mathbf{e}^{-\mathbf{S}} \in \mathbb{C}$ $\rightarrow$ **no probabilistic interpretation**

Model dependent solution: reformulation in terms of **dual variables**

Start with simple models and try to generalize!

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Dual Formulation – General Idea

Partition function Z $\in \mathbb{R}$

$$Z = \int \mathcal{D}[\phi]\, e^{-S[\phi]} \quad \text{with} \quad e^{-S} \in \mathbb{C} \quad \text{and} \quad \phi \ \ldots \ \text{“conventional fields”}$$

$\rightarrow$ try to find representation, such that

$$Z = \sum_{\{l\}} W[l] \quad \text{with} \quad W \in \mathbb{R}^+ \quad \text{and} \quad l \ \ldots \ \text{new degrees of freedom}$$

$\rightarrow$ probabilistic interpretation

$$\mathcal{P}[l] \equiv \frac{W[l]}{Z} \quad \ldots \quad \text{probability weight of configuration } l$$

ϕ^4 Model on the Lattice

Continuum action

$$S = \int d^4x \left[|\partial_\nu \phi|^2 + (m^2 - \mu^2)\,|\phi|^2 + \lambda\,|\phi|^4 + \mu\,(\phi^* \partial_4 \phi - \partial_4 \phi^* \phi) \right]$$

Lattice action

$$S = \sum_x \left(-\sum_{\nu=1}^{4} \left(e^{\mu\,\delta_{\nu,4}}\,\phi_x \phi^*_{x+\hat{\nu}} + e^{-\mu\,\delta_{\nu,4}}\,\phi^*_x \phi_{x+\hat{\nu}} \right) + \kappa\,|\phi_x|^2 + \lambda\,|\phi_x|^4 \right)$$

with kinetic, mass and self-interaction terms.

Chemical potential μ couples only in 4-direction (time).

$\rightarrow$ **Complex action problem** for $\mu \neq 0$.

Dual Formulation of the ϕ^4 Model – Sketch

Partition function

$$Z = \int \mathcal{D}[\phi]\ e^{-S} \sim \int \mathcal{D}[\phi] \prod_{x,\nu} e^{S_{x,\nu}} \sim \int \mathcal{D}[\phi] \prod_{x,\nu} \sum_{l_{x,\nu}=0}^{\infty} \frac{(S_{x,\nu})^{l_{x,\nu}}}{l_{x,\nu}!}$$

Integrating out the original fields ϕ_x, ϕ_x^* in terms of radial and angular parts ($\phi_x = r_x\ e^{i\,\theta_x}$)

→ **Kronecker deltas** (constrain the summation variables)

→ **Weight factors** (numerical integrals)

→ Partition function in dual representation

$$Z = \sum_{\{k,m\}} \prod_{x,\nu} \frac{1}{(|k_{x,\nu}| + m_{x,\nu})!\; m_{x,\nu}!} \quad \prod_x \delta\left(\sum_\nu [k_{x,\nu} - k_{x-\hat{\nu},\nu}]\right)$$
$$\times \prod_x e^{\mu k_{x,4}} \; \mathcal{W}\left(\sum_\nu \left[|k_{x,\nu}| + |k_{x-\hat{\nu},\nu}| + 2(m_{x,\nu} + m_{x-\hat{\nu},\nu})\right]\right)$$

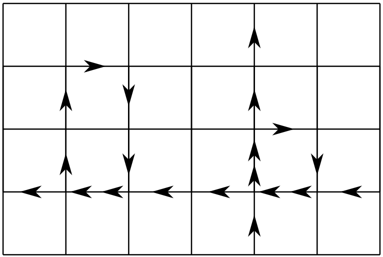
- **New degrees of freedom** (dual variables, fluxes)

  $k_{x,\nu} \in \mathbb{Z}$ (constrained) and $m_{x,\nu} \in \mathbb{N}_0$ (unconstrained)
- **Weight factors** $\mathcal{W}(n_x)$ (numerical integrals)
- **Complex action problem is gone**

  $$Z = \sum_{\{k,m\}} P[k,m] \quad \text{with} \quad P[k,m] \in \mathbb{R}^+ \sim \text{probability weight}$$
- **Exact rewriting**

Numerical Simulation

constraint $\Longleftrightarrow$ **flux conservation** at each lattice site (**closed loops**)



$\rightarrow$ Generalization of the **Prokof'ev-Svistunov worm algorithm**

- Standard Metropolis update sweep for the m-variables
- Worm update for the restricted k-variables

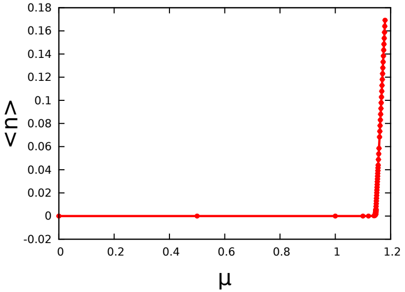
Results

Particle number density

$$\langle n \rangle \propto \frac{\partial \ln Z}{\partial \mu} \propto \left\langle \sum_x k_{x,4} \right\rangle \quad \sim \quad \text{winding number}$$

The Silver Blaze Phenomenon

At $T = 0$ physics is independent of $\mu < \mu_{\text{crit}}$.



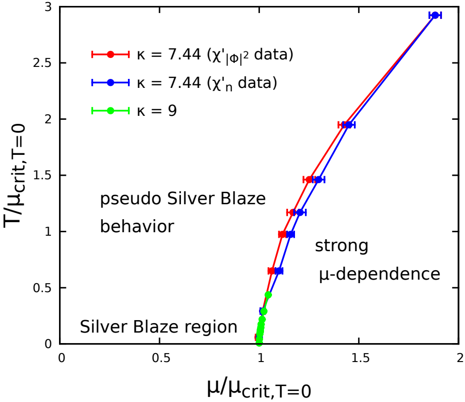
second order phase transition at

$$\mu_{\text{crit}} = m_{\text{ren}} = 1.146(1)$$

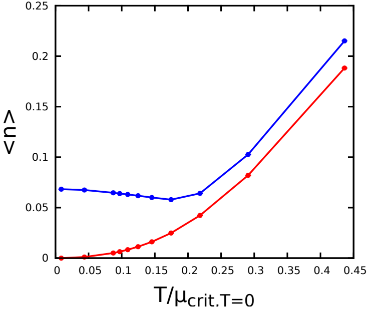
$$\kappa = 9, \ \lambda = 1, \ T = 0$$

Analysis at different temperatures (and observables, parameter sets, ...)

→ **Phase diagram** and **relativistic Bose condensation**



Vertical sections below and above μ_{crit}.



2-Point Functions

Correlators

$$\langle \phi_a \, \phi_b^* \rangle = \frac{1}{Z} \int \mathcal{D}[\phi] \; e^{-S} \; \phi_a \; \phi_b^* \equiv \frac{1}{Z} \; Z_{a,b}$$

cannot be expressed as partial derivatives of Z.

Dual representation of $Z_{a,b}$

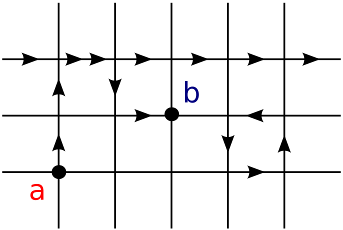
$$\sum_{\{k,m\}} \prod_x \delta\left(\sum_\nu \left[\dots \right] - \delta_{x,a} + \delta_{x,b} \right) \mathcal{W}\left(\sum_\nu \left[\dots \right] + \delta_{x,a} + \delta_{x,b} \right) \left(\dots \right)$$

$\rightarrow$ modified arguments of

- Kronecker deltas and
- Weightfactors at lattice sites a and b.

Numerical Simulation

New Constraint for $Z_{a,b}$

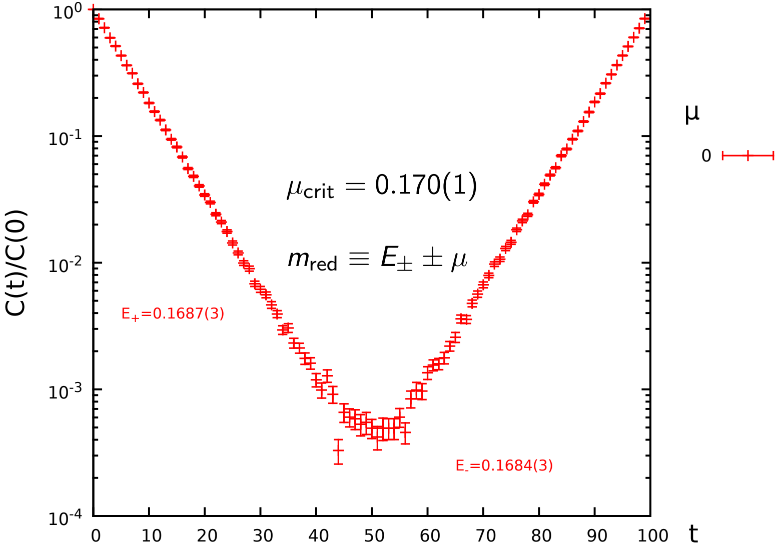


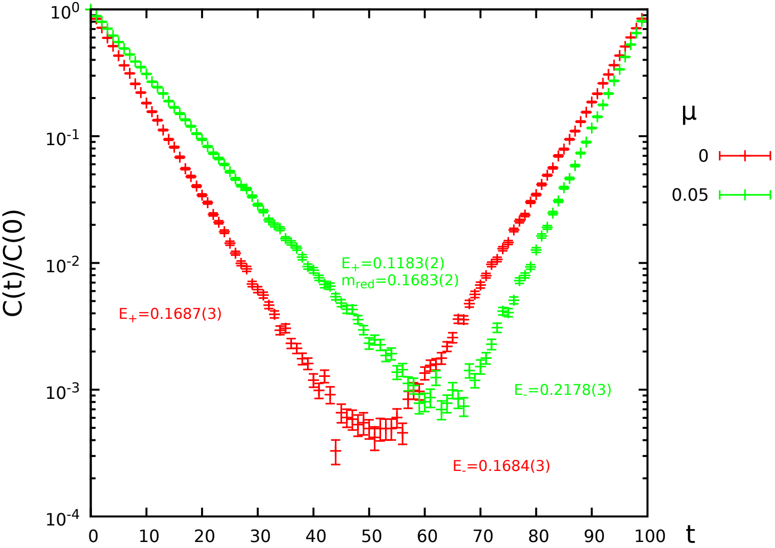
Enlarged ensemble (Korzec et. al., Comput. Phys. Commun. **182** (2011))

$$\mathcal{Z} \equiv \sum_{a,b} Z_{a,b} \quad \sim \quad \text{closed loops } (a = b) \; + \; \text{open lines } (a \neq b)$$

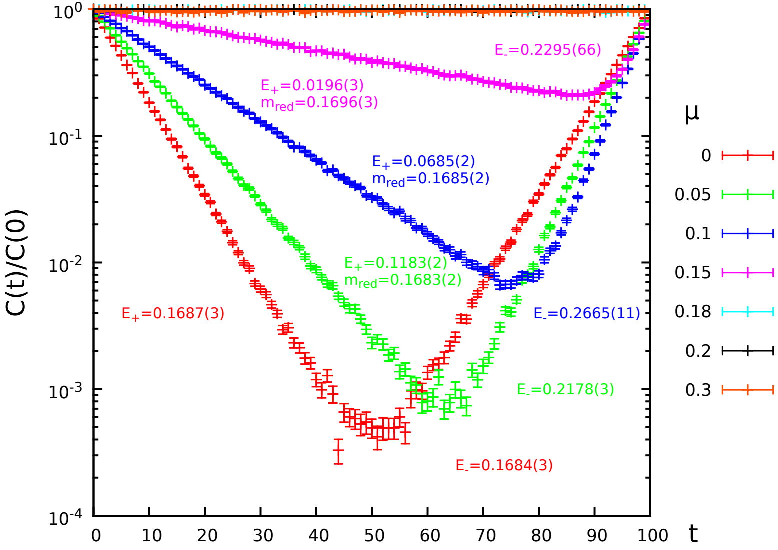
Dual zero momentum temporal correlator

$$C(t) \;\propto\; \langle \delta_{t\,,\,a_4 - b_4} \rangle_{\mathcal{Z}} \;\propto\; e^{-m\,t}$$


$\mu_{crit} = 0.170(1)$
$m_{red} \equiv E_{\pm} \pm \mu$
E_+=0.1687(3)
E_-=0.1684(3)
C(t)/C(0)
t
μ
0
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0 10 20 30 40 50 60 70 80 90 100



C(t)/C(0)
t
μ
0
0.05
$E_+=0.1183(2)$
$m_{red}=0.1683(2)$
$E_+=0.1687(3)$
$E_-=0.2178(3)$
$E_-=0.1684(3)$



Fits to the slopes behave exactly as expected!

Conclusion

- Different models can be mapped to a dual representation $\longrightarrow$ **Complex action problem solved**.
- **Physical Observables** (e. g. particle number) and properties (e. g. phase diagram) can be studied **in terms of dual variables**.
- Generalization allows us to carry out **spectroscopy** calculations for **non-zero chemical potential**.

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Outlook

- Continue work on correlators.
- **Find dual formulation for models with non-abelian degrees of freedom.**
- At the moment: SU(2) spin model

$$Z = \int \mathcal{D}[U] \prod_{x,\nu} e^{\beta \, \mathrm{Tr}[U_x U^\dagger_{x+\hat{\nu}}] \, + \, h \, \mathrm{Tr}[U_x]} \qquad U_x \in SU(2)$$

successfully reformulated.