Quark Number Susceptibility Divergence can be Subtracted Off

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Introduction

♠ Introducing chemical potential by adding $a\mu\bar{\psi}(x)\gamma_0[\psi(x+\hat{0}) + \psi(x-\hat{0})]$ leads to a^{-2} divergences in energy density and quark number susceptibility.
♠ In full QCD, this amounts to weights $f(a\mu) = 1 + a\mu$ and $g(a\mu) = 1 - a\mu$ to forward and backward time links respectively.

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♡ Important to note that analytical proof was for *free* quarks in all these cases; Numerical computations showed it to work for the interacting case (Gottlieb et al. PRL 1987, RVG,Potvin,Sanielevici PRD 1988).

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♡ Using the definition of the chiral projectors for overlap fermions, we (Gavai-Sharma, PLB 2012) proposed a chirally invariant Overlap action for nonzero μ :

$$
\begin{aligned}
S^F &= \sum_n [\bar\psi_{n,L}(aD_{ov} + a\mu\gamma^4)\psi_{n,L} + \bar\psi_{n,R}(aD_{ov} + a\mu\gamma^4)\psi_{n,R}] \\
&= \sum_n \bar\psi_n [aD_{ov} + a\mu\gamma^4(1 - aD_{ov}/2)]\psi_n \ .
\end{aligned}
$$

- Easy to check that under the chiral transformations, $\delta\psi = i\alpha\gamma_5(1 - aD_{ov})\psi$ and $\delta\bar{\psi} = i\alpha\bar{\psi}\gamma_5$, it is invariant or all values of $a\mu$ and a.

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- Order parameter exists for all μ and T. It is $\langle\bar{\psi}\psi\rangle = \lim_{am\to 0} \lim_{V\to\infty} \langle \mathrm{Tr}\ \frac{(1-aD_{ov}/2)}{[aD_{ov}+(am+a\mu\gamma^4)(1-aD_{ov}/2)]} \rangle$.

- It, however, has a^{-2} divergences which cannot be removed by exponentiation of the μ-term (Narayanan-Sharma, JHEP 2011).

Tackling the Divergences

- Note that contrary to common belief, divergences are **NOT** a lattice artifact. Indeed lattice regulator simply makes it easy to spot them. Using a Pauli-Villars cut-off Λ in the continuum theory, one can show the presence of $\mu\Lambda^2$ terms in number density easily.

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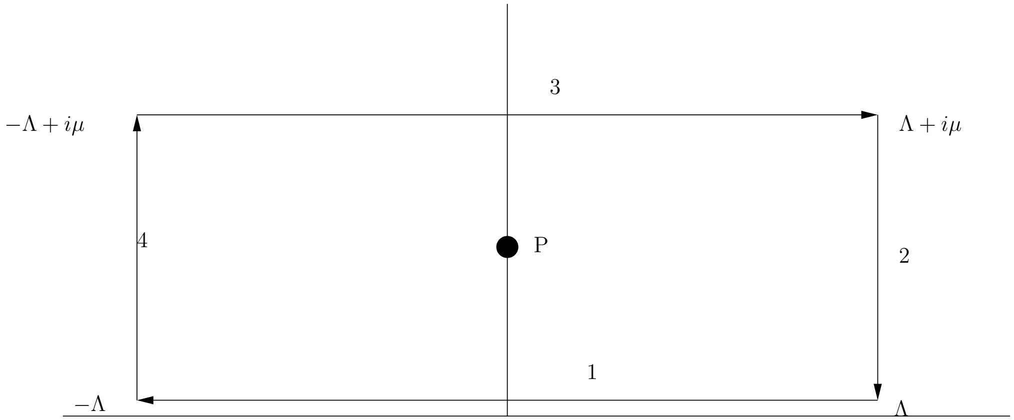
- The expression for the number density is

$$n = \frac{2iT}{V} \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{(\omega_n - i\mu)}{f + (\omega_n - i\mu)^2} \equiv \frac{2iT}{V} \sum_n \int \frac{d^3p}{(2\pi)^3} \sum_{\omega_n} F(\omega_n, \mu, \vec{p}), \quad (1)$$

where $f = p_1^2 + p_2^2 + p_3^2$. Here we take the gamma matrices as all Hermitian.

- Vaccum contribution is removed by subtracting $n(T = 0, \mu = 0)$.

- In the usual contour method, but with a cut-off Λ, one has in the $T \to 0$ limit but $\mu \neq 0$ the following :



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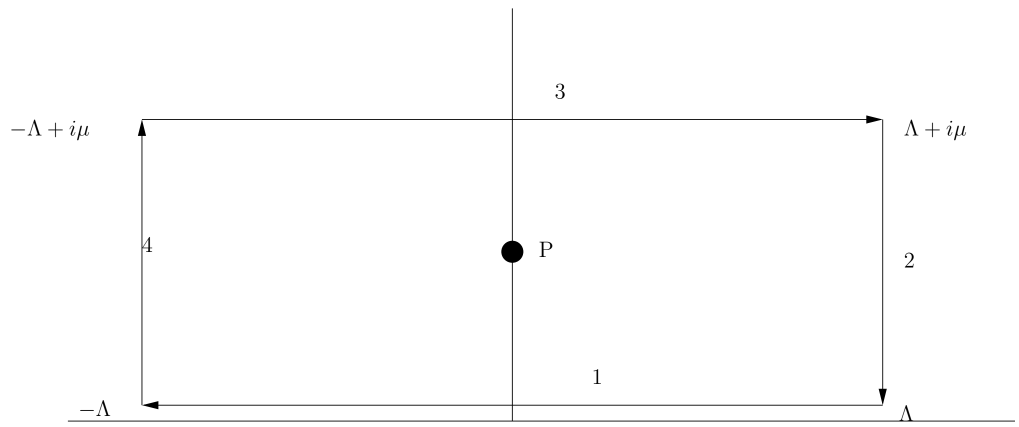

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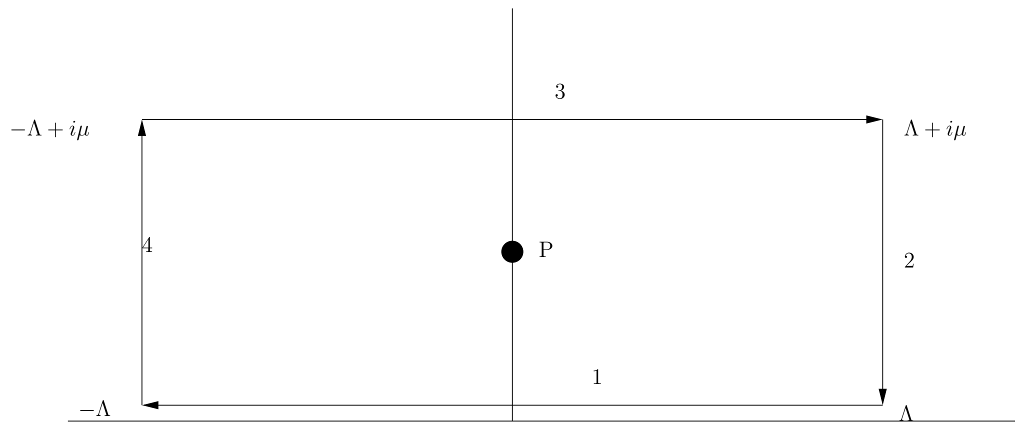

- The $\mu\Lambda^2$ terms arise from the arms 2 & 4 in figure above.

- One may thus follow the prescription of subtracting the free theory divergence by hand. If it works, one can have several computational advantages in computing the higher order susceptibilities needed in critical point search.

- Indeed, it can be used for the staggered fermions as well, leading to $M' = \sum_{x,y} N(x,y)$, and $M'' = M''' = M''''... = 0$,
  in contrast to the popular exp($\pm a\mu$)-prescription where *all* derivatives are nonzero:
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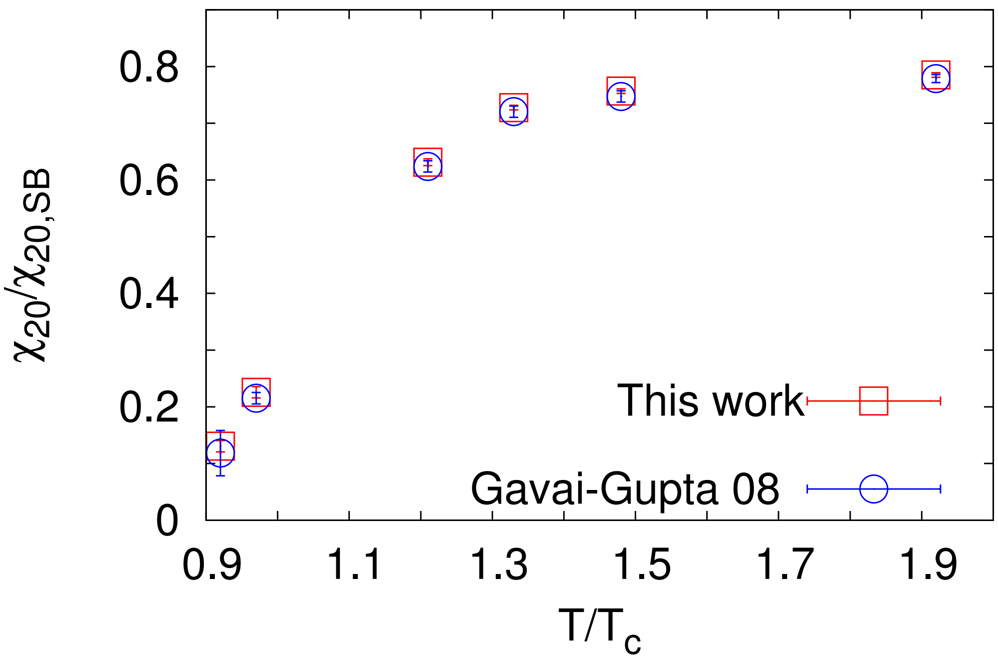
- Lot fewer terms in the Taylor coefficients, especially as the order increases. E.g., in the 4th order susceptibility, $\mathcal{O}_4 = -6 \text{ Tr } (M^{-1}M')^4$ in our proposal, compared to $\mathcal{O}_4 = -6 \text{ Tr } (M^{-1}M')^4 + 12 \text{ Tr } (M^{-1}M')^2 M^{-1}M'' - 3 \text{ Tr } (M^{-1}M'')^2 - 3 \text{ Tr } M^{-1}M'M^{-1}M''' + \text{Tr } M^{-1}M''''$.

- $\mathcal{O}_8$ has one term in contrast to 18 in the usual case. $\Longrightarrow$ Number of M^{-1} computations needed are lesser.

Testing the idea

- On our $N_t = 6$ configurations (Gavai-Gupta PRD 2008), where we computed and published all the coefficients, the proposal of linear μ with simple subtraction was tested (Gavai-Sharma PRD 2012).



- Since the corresponding free fermion results approach the continuum limit differently, the $N_t = 6$ free results were divided out above.

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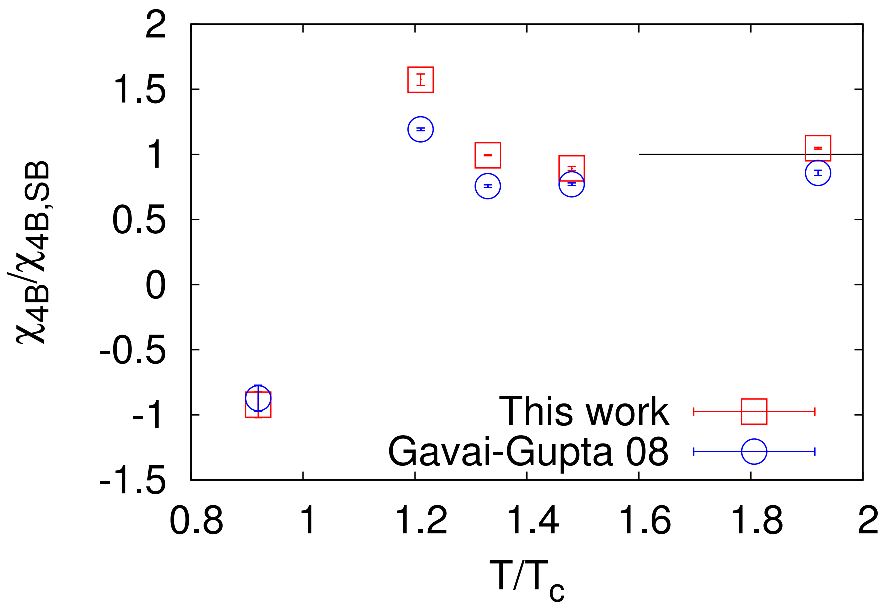
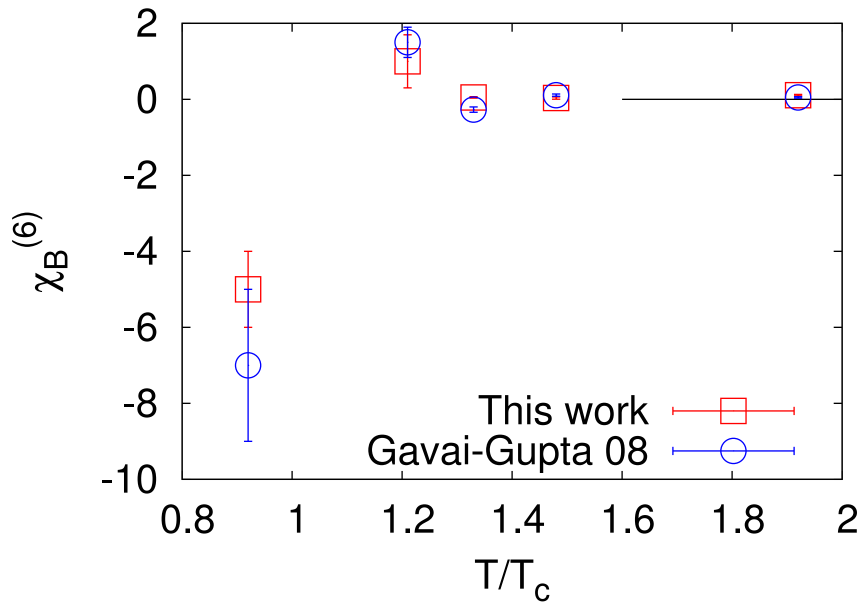
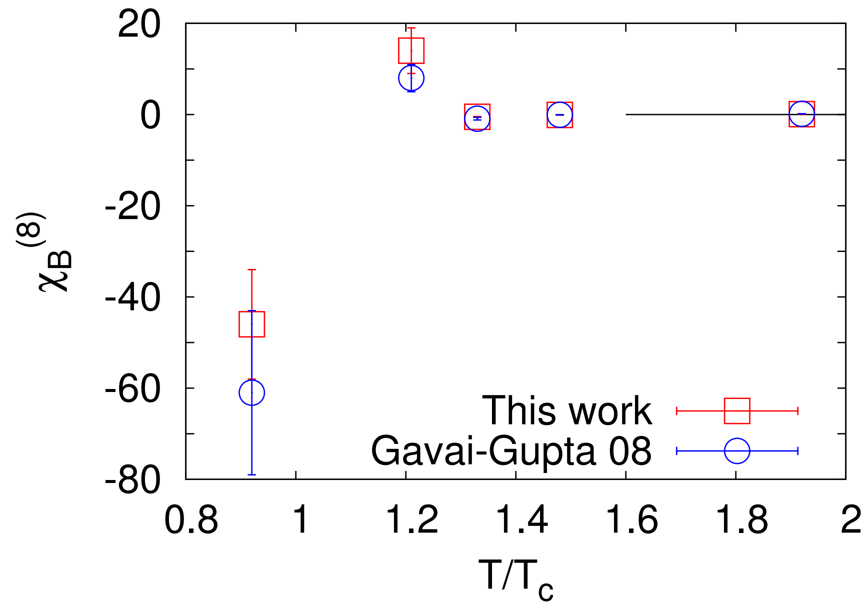
- Our 4th order results are







- A comparison of the 6th and 8th order is displayed above. All these results suggest a reasonable agreement with the usual action.

- In order to test whether the divergence is truely absent, one needs to take the continuum limit $a \to 0$ or equivalently $N_t \to \infty$.

- We decided to test it for quenched QCD for which configuration generation for larger N_t is simpler.

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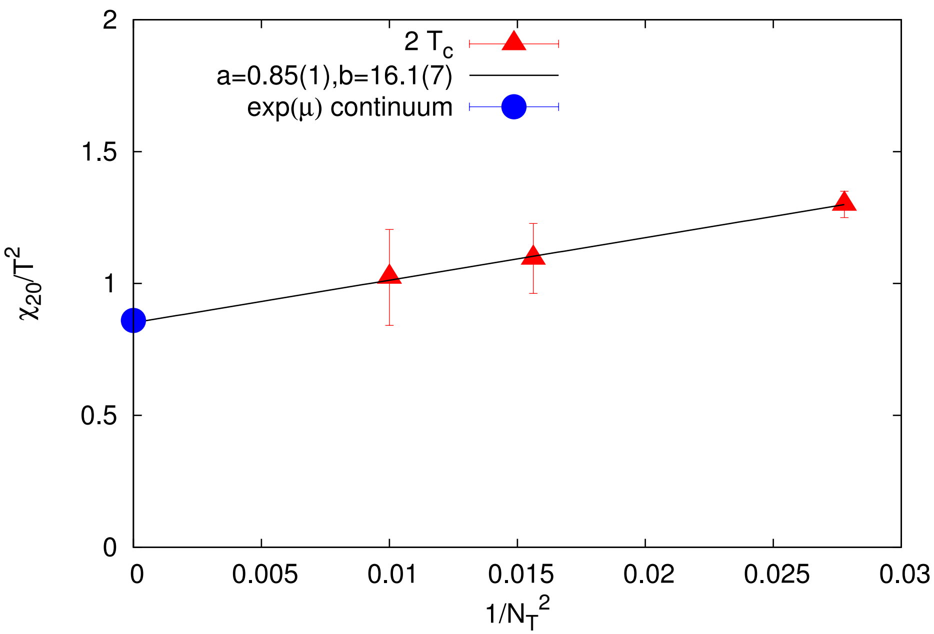
- For $m/T_c = 0.1$, we employed $N_t = 6$, 8 and 10 lattices and for 50-80 independent configurations. At $T = 2T_c$, we obtained



- Absence of any divergent term is evident in the positive slope of the data.

- Moreover, our extrapolated continuum result coincides with the earlier result obtained with the $\exp(\pm a\mu)$ action (Swagato Mukherjee PRD 2006).

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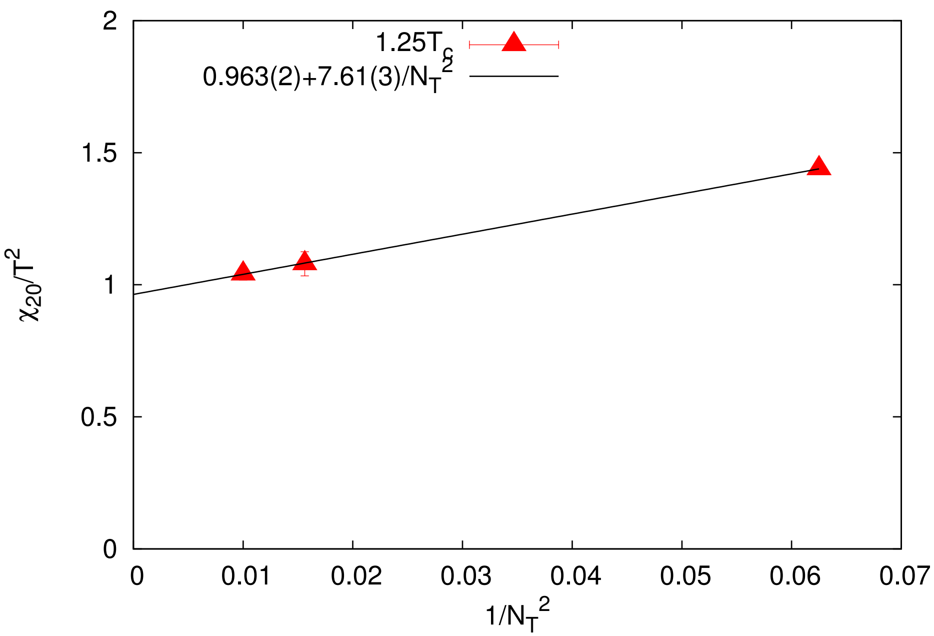
- We lowered the mass by a factor on 10 to $m/T_c = 0.01$ & repeated the exercise at a lower temperature on $T/T_c = 1.25$.



- Again no divergent term is evident in the slope suggested by the data.

Summary

- Actions linear in μ can be employed safely, and may have computational advantages.
- Divergence in the quark number susceptibility can be subtracted off by the corresponding free theory result. Continuum extrapolation yields the same result for both the linear and the exponential form, as it must.

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- Divergence in the quark number susceptibility can be subtracted off by the corresponding free theory result. Continuum extrapolation yields the same result for both the linear and the exponential form, as it must.

- Interactions do not induce any additional divergence at finite T or μ once all zero temperature divergences are removed. This has been well known perturbatively but seems to hold non-perturbatively as well.