Non-perturbative approaches to transport properties & spectral functions in hot QCD

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SFB 1044 THE LOW-ENERGY FRONTIER OF THE STANDARD MODEL

PRISMA

HIM

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Thermal Field Theory: basic relations

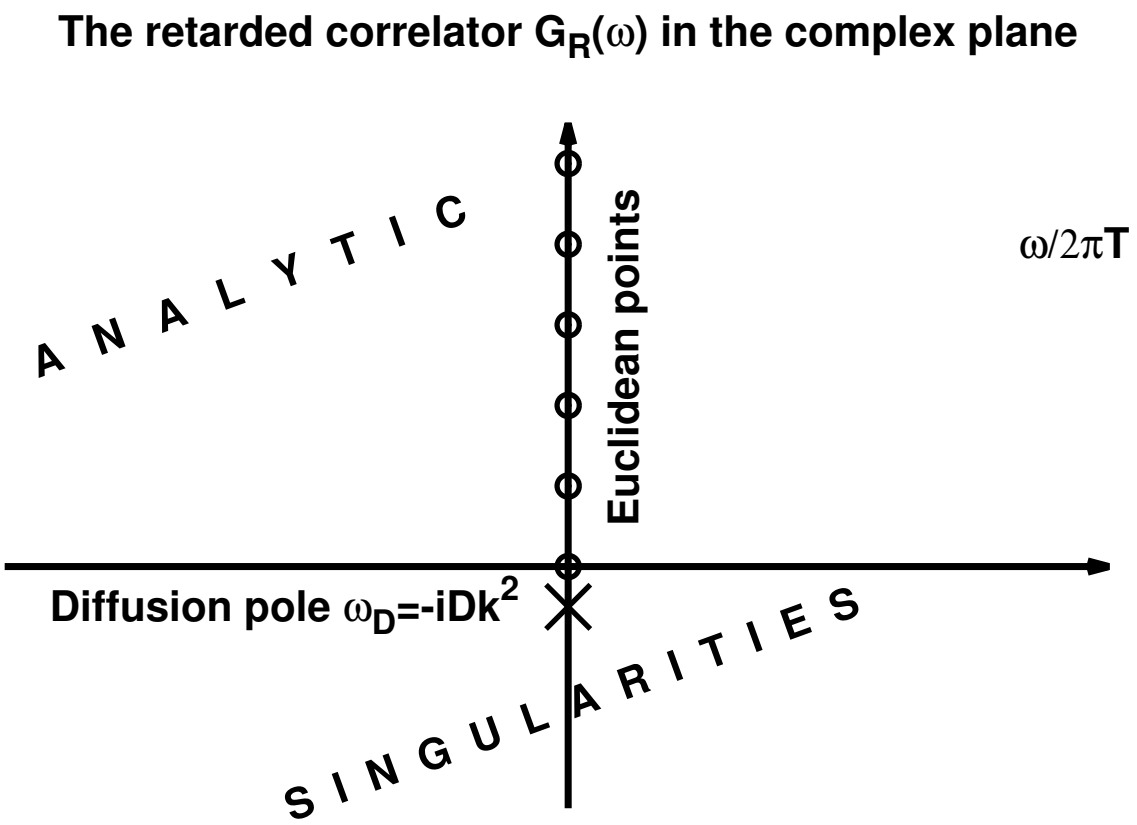
- $\langle A(t)\rangle = \mathrm{Tr}\,\{\hat{\rho}\,A(t)\}, \qquad \hat{\rho} = \frac{e^{-\beta H}}{Z}, \quad A(t) \equiv e^{iHt}A(0)e^{-iHt}.$
- Wightman correlator: $G_>(t) \equiv \langle A(t)\,B(0)\rangle$
- Euclidean correlator: $G_E(t) \equiv G_>(-it)$
- $G_R(\omega) \equiv i\int_0^\infty dt\; e^{i\omega t}\langle[A(t), B(0)]\rangle$
- $\tilde{G}_E(\omega_\ell) = G_R(i\omega_\ell), \quad \omega_\ell = 2\pi T\ell > 0.$
- spectral function:

$$\rho(\omega) = \frac{1}{2\pi i}\left(G_R^{AB}(\omega) - G_R^{B^\dagger A^\dagger}(\omega)^*\right) \overset{B=A^\dagger}{=} \frac{1}{\pi}\mathrm{Im}\,G_R(\omega).$$

- relation between Euclidean correlator and the spectral function:

$$\begin{aligned} G_E(\tau) + G_E(\beta-\tau) &= \int_{-\infty}^{\infty} d\omega\,\rho(\omega)\,\frac{\cosh\omega(\frac{1}{2}\beta - t)}{\sinh\frac{1}{2}\beta\omega}, \\ \tilde{G}_E(\omega_\ell) &= \int_{-\infty}^{\infty} d\omega\,\frac{\omega\,\rho(\omega)}{\omega^2+\omega_\ell^2}, \quad \text{(under conditions)}. \end{aligned}$$

Illlustration for two-point function of $\tilde{j}_0(\boldsymbol{k}) = \int d^3\boldsymbol{x}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\, j_0(\boldsymbol{x})$



Carlson's theorem $\Rightarrow$
if $G_R(\omega)$ is continuous for $\mathrm{Im}(\omega) \geq 0$,
the Euclidean points uniquely determine $G_R(\omega)$ for $\mathrm{Im}(\omega) \geq 0$.

HM 1104.3708

Explicit inversion of the correlator

The retarded correlator $G(t) \equiv i\langle[A(t), B(0)]\rangle$ can be obtained as a series in Laguerre polynomials

$$\begin{aligned} G(t) &= \exp\big[-\exp(-2\pi T t)\big] \sum_{\ell=0}^{\infty} a_\ell \, L_\ell(2e^{-2\pi T t}), \\ a_\ell &\equiv 2(-1)^\ell \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} G_E(\omega_{n+1}) \, {}_2F_1(-\ell, n+1; 1; 2) \, . \end{aligned}$$

${}_2F_1$ is the hypergeometric function. Valid if $G_E(\tau)$ is continuous at $\tau = 0$.

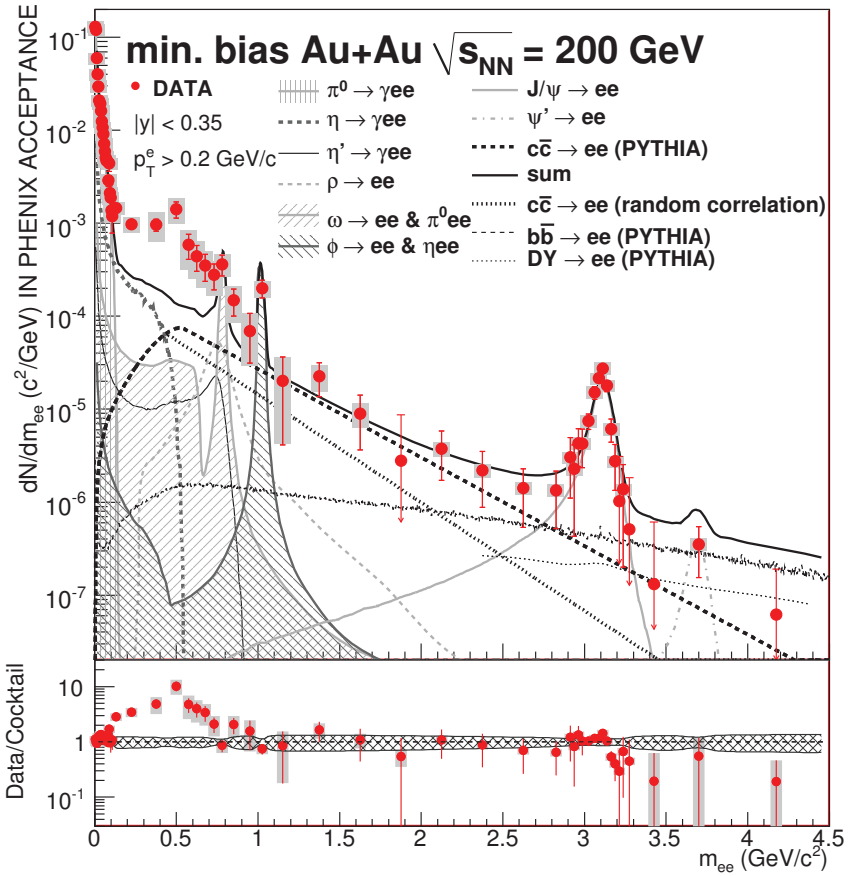
NB.

- $G(t) \sim \chi_s D\boldsymbol{k}^2 \exp(-D\boldsymbol{k}^2 t)$ for a channel exhibiting diffusion (until non-linear effects become important [Kovtun, Moore, Romatschke, 1104.1586])
- here, $\lim_{t\to\infty} G(t) = \sum_{\ell=0}^{\infty} a_\ell$

[G. Cuniberti, E. De Micheli, G.A. Viano, Commun. Math. Phys. 216, 59 (2001); Burnier, Laine, Mether, 1101.5534]

Dilepton spectrum in Au-Au at RHIC measured by PHENIX

Phys. Rev. C 81, 034911 (2010)

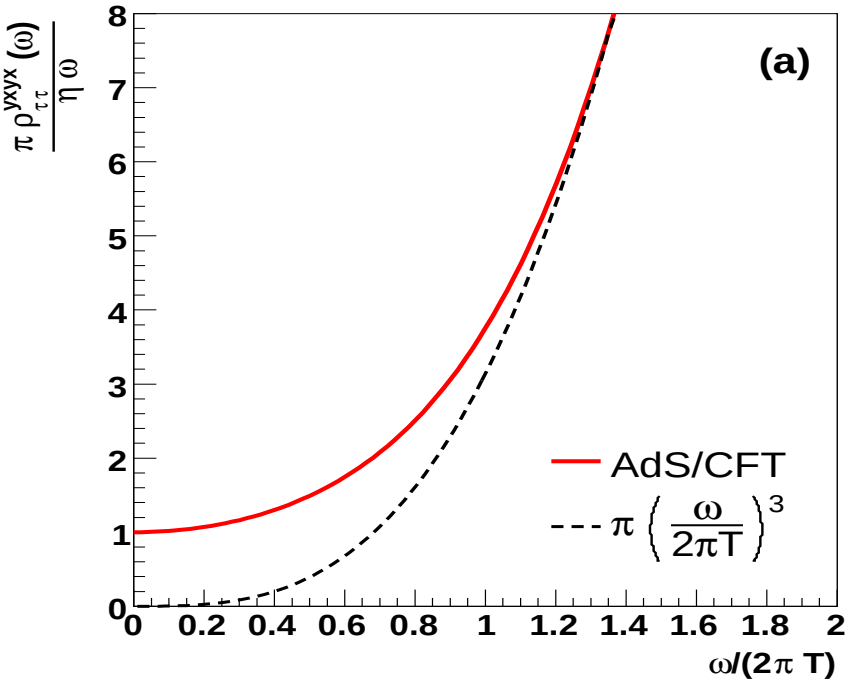


thermal production rate of dilepton pairs of invariant mass $M^2 = \omega^2 - \boldsymbol{k}^2$:

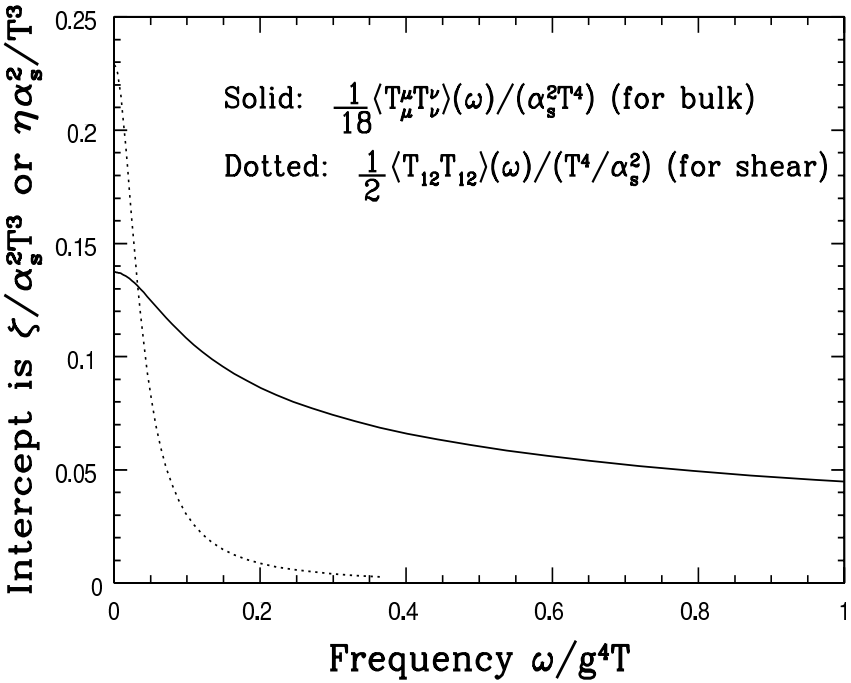
$$\frac{\mathrm{d}N_{\ell_+\ell_-}}{\mathrm{d}\omega\,\mathrm{d}\boldsymbol{k}^3} = \left(\sum\nolimits_{\mathrm{f}} Q_{\mathrm{f}}^2\right) \frac{\alpha_{\mathrm{em}}^2}{6\pi^3} \frac{\rho_{\mu\mu}(\omega, \boldsymbol{k}, T)}{(\omega^2 - \boldsymbol{k}^2)(e^{\omega/T} - 1)}.$$

Shear channel: weak vs. strong coupling

RHIC & LHC: large observed elliptic flow + hydrodynamic calculations $\rightsquigarrow$ $\eta/s \lesssim 0.5$. Is this consistent with a quasiparticle picture of the QGP?



AdS/CFT [Teaney 06]



Weak coupling [Moore, Saremi 2008]

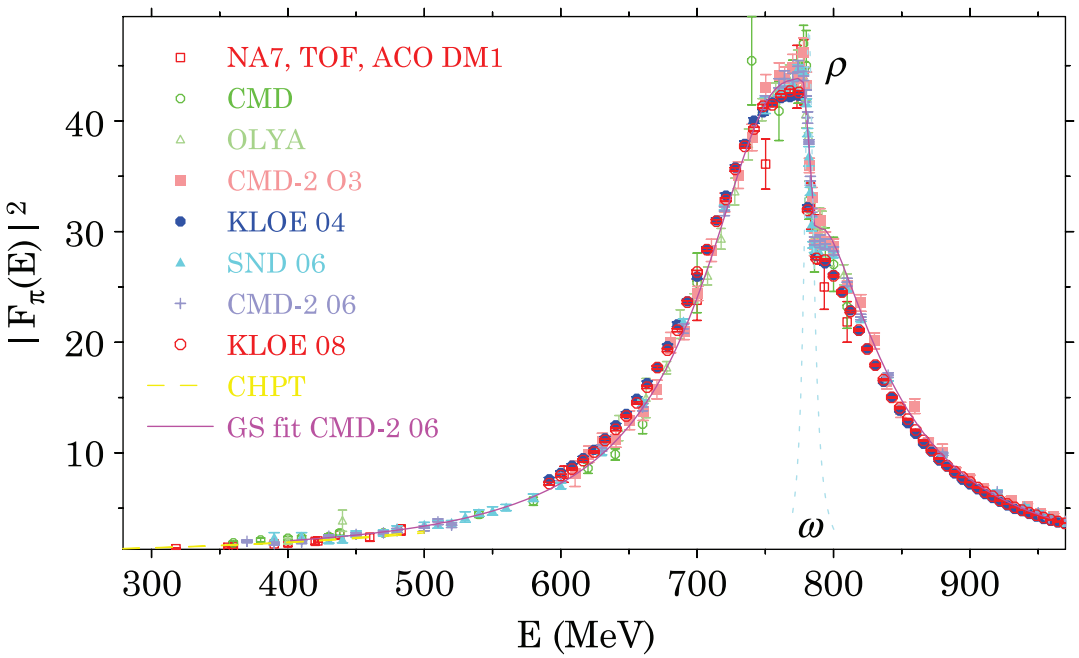
Q: is the shear QCD spectral function like the red or like the dotted curve?

Similar questions in the unitary Fermi gas: e.g. [Wlazlowski, Magierski, Bulgac, Roche 1304.2283].

Real-time dynamics & lattice QCD

- lattice QCD has delivered the equation of state ot satisfactory precision
- spectral functions? solving integral equation = numerically ill-posed probl.
- at $T = 0$, $\exists$ a direct recipe for the low-lying part of the spectral function.

ρ channel: $\rho(q^2) = \frac{1}{48\pi^2}\left(1 - 4m_\pi^2/q^2\right)^{3/2} |F_\pi(q^2)|^2$ up to $q^2 < (4M_\pi)^2$



Time-like pion form factor:

$${}_{\text{out}}\langle \pi_+\pi_- | \boldsymbol{j} | 0 \rangle = (\boldsymbol{p}_+ - \boldsymbol{p}_-) F_\pi(q^2)$$

[plot from Jegerlehner, Nyffeler 0902.3360]

$$\boxed{|F_\pi(q^2)|^2 = \left(q\phi'(q) + k\frac{\partial\delta_1(k)}{\partial k}\right)\frac{3\pi q^2}{2k^5 L^3}\left|{}_L\left\langle \pi\pi \left| \int \mathrm{d}\boldsymbol{x}\, j^z(\boldsymbol{x}) \right| 0 \right\rangle\right|^2}$$ [HM 1105.1892]

Current non-perturbative approaches to spectral functions

In some cases, reexpress the quantity through static quantities:

- speed of sound $c_s^2 = \partial p / \partial e$
- pion dipersion relation below T_c [Son-Stephanov 02]

'Improve' the problem:

- sum rules
- reduce ultraviolet contributions
- use effective theory to eliminate a scale from the problem

Relatively new in this field:

- functional methods [J. Pawlowski at Confinement '12]

Radically new:

- quantum simulation of lattice gauge theories (see U. Wiese 1305.1602).

I will cover

- a lattice QCD calculation of the isovector vector channel in the QGP
- thermal sum rules
- static second order hydrodynamic coefficients.

Other developments covered in dedicated talks:

- NRQCD studies of quarkonium ($\rightarrow$ talk by S. Kim)
- Qhat ($\rightarrow$ K. Rummukainen)

The $I = 1$ vector channel [Brandt, Francis, HM, Wittig 1212.4200; Burnier, Laine 1201.1994]

LO perturbative prediction:

$$\rho(\omega, T) = 2\pi\chi_s\langle v^2\rangle\omega\delta(\omega) + \frac{N_c}{2\pi}\theta(\omega - 2m)\left[1 - \frac{4m^2}{\omega^2}\right]^{\frac{1}{2}}\left[1 + \frac{2m^2}{\omega^2}\right]\omega^2 \tanh\frac{\omega}{4T}.$$

Introduce $G^{rec}(\tau, T) =$ Euclidean correlator that would be realized at temp. T if the spectral function was unchanged between temperature T and $T = 0$.

Compute it via $G^{\rm rec}(\tau, T) = \sum_{m\in\mathbb{Z}} G(|\tau + m\beta|, T = 0)$.

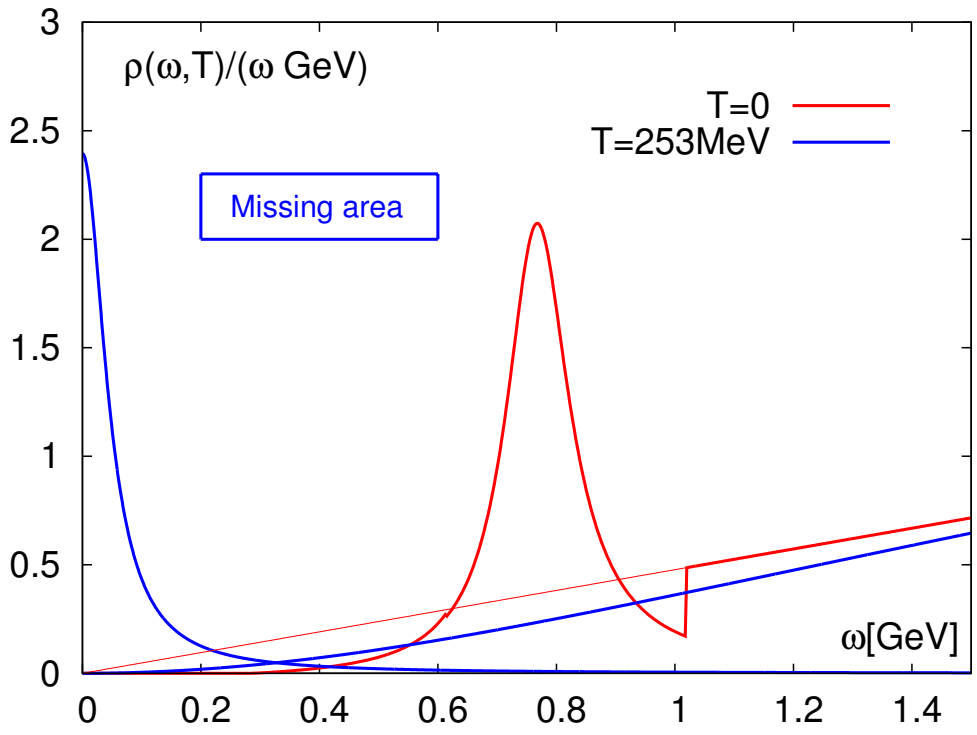
Method based on [HM, 1002.3343]

$$\underbrace{[G(\tau) - G^{rec}(\tau)]}_{\text{from Lattice QCD}} = \int_0^\infty \frac{d\omega}{2\pi}\underbrace{(\rho(\omega, T) - \rho(\omega, T = 0))}_{\text{make Ansatz for difference}}\frac{\cosh[\omega(\beta/2 - \tau)]}{\sinh(\omega\beta/2)}$$

Sum rule: $\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\ [\rho(\omega, T) - \rho(\omega, 0)] = 0.$

[Bernecker, HM 1107.4388]

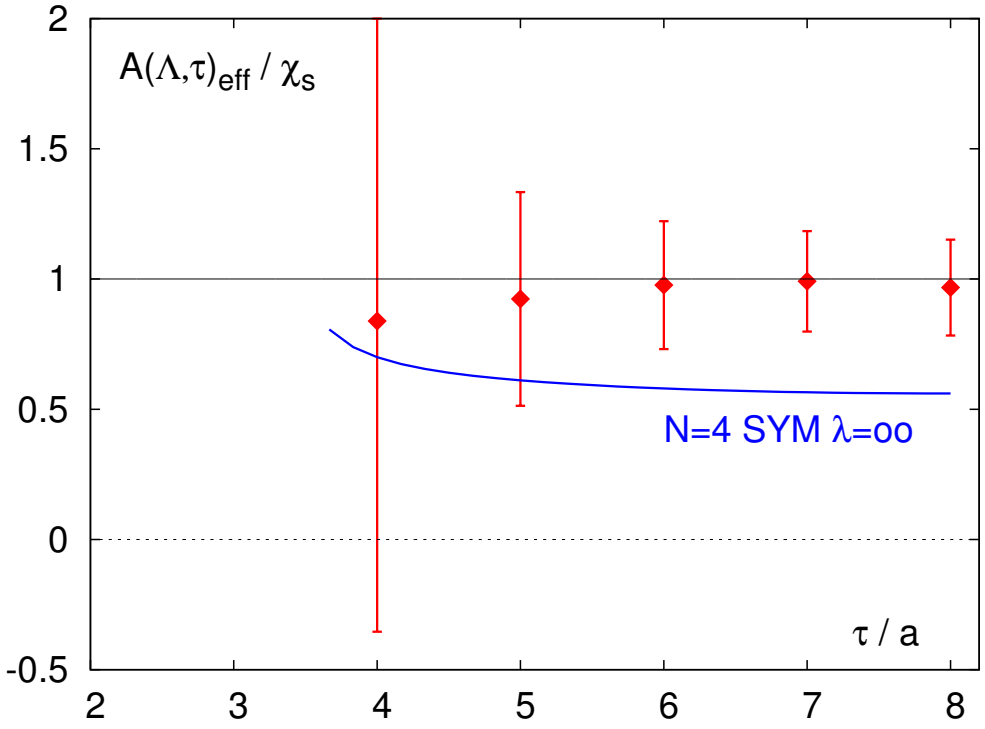
Sum rule & phenomenology of the vector channel



- phenomenological isovector vector spectral function in the vacuum
- finite-T: weak-coupling spectral function with $\langle v^2 \rangle = 1$,
- $\chi_s / \chi_s^{\rm SB} = 0.88(1)$
- sum rule: thermal spectral function must receive additional contributions of the size given by the 'missing area' rectangle.

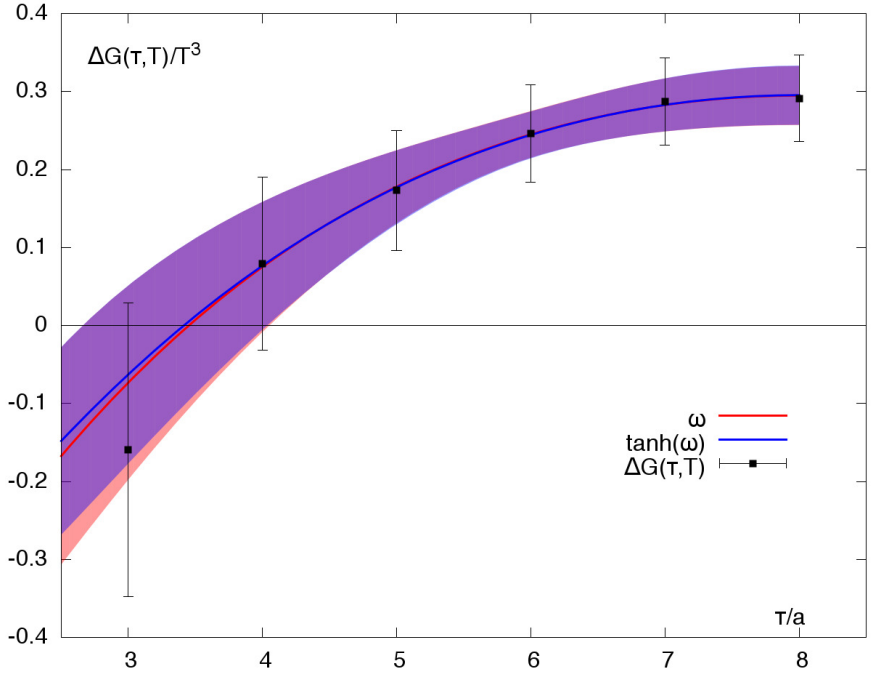
A ridiculous Ansatz

- $\frac{\rho(\omega,T)}{2\pi\omega} = A(\delta(\omega) - \frac{1}{2}\delta(\omega - m_1) - \frac{1}{2}\delta(\omega + m_1))$, $m_1 = \rho$-meson mass
- $A_{\rm eff}(t) = \frac{\Delta G(t,T)}{T - m_1/2\cosh[m_1(\beta/2-t)]/\sinh[m_1\beta/2]}$

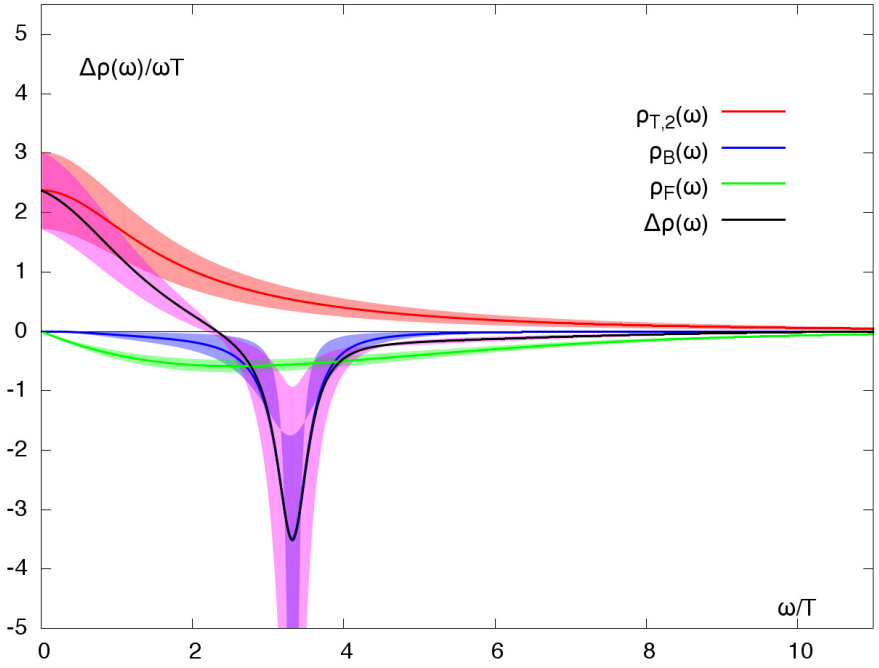


A smooth Ansatz

Lattice QCD correlator



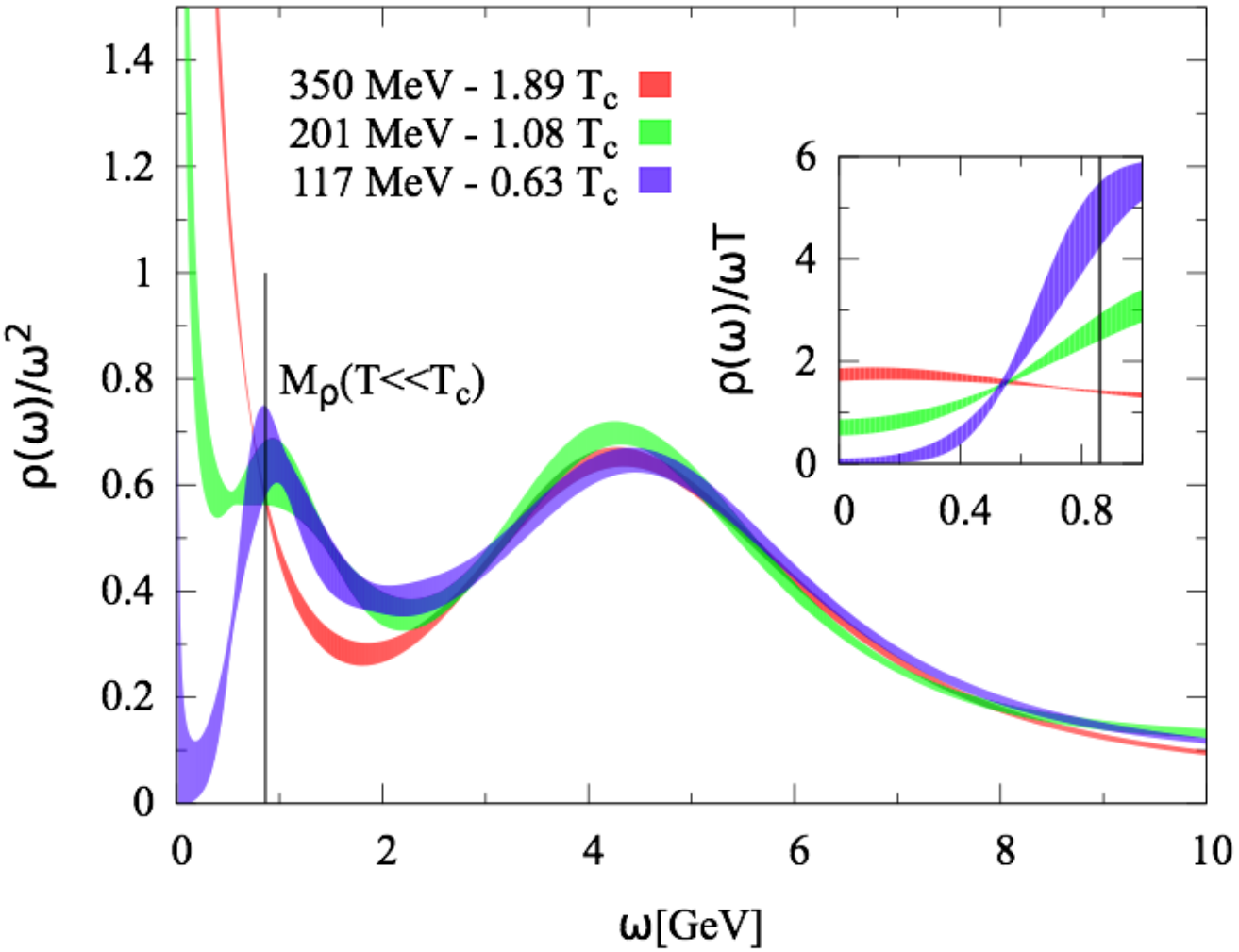
Spectral function



- first calculation of this channel with dynamical quark effects
- $T = 255\text{MeV}, \;\; m_\pi = 270\text{MeV}, \;\; 16 \times 64^3$

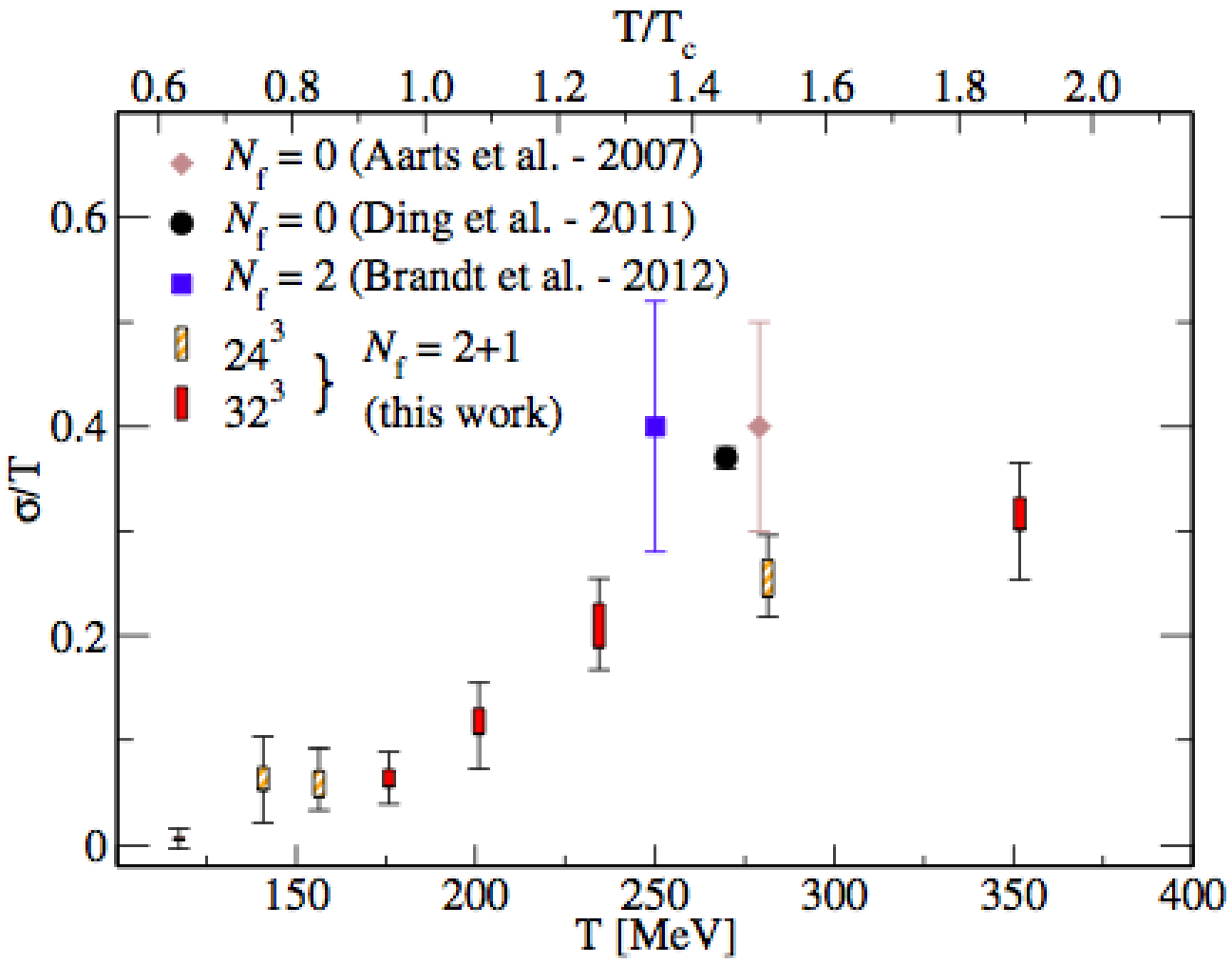
A. Francis, B. Brandt, HM, H. Wittig JHEP 1303 (2013) 100

Scan in temperature [Aarts et al. 1307.6763]



MEM reconstruction.

Isospin conductivity: summary [Aarts et al. 1307.6763]



Thermal sum rules

- Kapusta-Shuryak '93: $\rho_V - \rho_A$
  (generalized Weinberg sum rules to finite-T)
- Kharzeev-Tuchin '07: $\int_0^\infty \frac{d\omega}{\omega}\, \Delta\rho_\theta(\omega, \mathbf{0}, T) = T^5 \frac{\partial}{\partial T} \frac{e-3p}{T^4}$
- Romatschke-Son '09: shear sum rule in N=4 SYM
  (left unfinished in pure Yang-Mills).
- Bernecker & HM '11: $\int_0^\infty d\omega\, \omega\, \Delta\rho_{nn}(\omega, \boldsymbol{q}, T) = 0 \quad \forall \boldsymbol{q}$
  (n= charge density of the vector current)

From lattice practitioner' point of view: provide constraints on the spectral function in terms of expectation value of a local operator.

Sum rules for the EMT in SU(N) gauge theory Giusti, HM, in prep

Contact terms of the conserved charges: the WIs of translation invariance imply

$$\int d^4x\, e^{iq_0x_0}\langle T_{00}(x)T_{00}(0)\rangle = \left\langle T_{00} + \tfrac{1}{4}F^2\right\rangle.$$

$$\int d^4x\, e^{iq_0x_0}\langle T_{03}(x)T_{03}(0)\rangle = \left\langle T_{33} - F^a_{03}F^a_{03} + \tfrac{1}{4}F^2\right\rangle.$$

Consider the linear combination

$$\mathcal{T}(q_0) = \int d^4x\, e^{iq_0x_0}\left(e^{iq_1x_1}T_{13}(x)T_{13}(0) - \tfrac{2}{3}T_{00}(x)T_{00}(0)\right).$$

$q_0 = 0$: From the WIs, one shows that

$$\langle\mathcal{T}(0)\rangle_T - \langle\mathcal{T}(0)\rangle_0 = -\frac{2c_v}{3L_0} + \frac{1}{2}(e+p) + \frac{1}{9}(e-3p).$$

$q_0 \to \infty$: in rotation symmetric state, only two linearly independent operators. The Wilson coefficient of O_{e+p} in momentum-space OPE is of order $g^2(\omega)$ [CaronHuot 2009 , HM 2010, Schröder et al 2011].
Wilson coefficient of $T_{\mu\mu}$: computed to two-loop order by [Chetyrkin, Zoller 1209.1516].

Sum rules for the EMT in SU(N) gauge theory (II)

Result:

$$\lim_{\omega\to\infty} \Delta\mathcal{T}(\omega) = c\cdot(-T_{\mu\mu}), \qquad c = -\frac{4}{99}.$$

After subtracting the contact term, the spectral representation reads

$$\langle \mathcal{T}(0) - \lim_{\omega\to\infty} \mathcal{T}(\omega)\rangle_{T-0} = \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \left(\Delta\rho_{1313}(\omega, q_1, T) - \frac{2}{3}\Delta\rho_{0000}(\omega, T) \right).$$

Final result: valid for any value of the spatial momentum q_1.

$$\int_{-\infty}^{\infty} \frac{d\omega}{\omega}\, \Delta\rho_{1313}(\omega, q_1, T) = \frac{1}{2}(e+p) + \frac{5}{33}(e-3p).$$

Shear and sound sum rules: summary

Giusti, HM, in prep

$$
\begin{aligned}
-\int_{-\infty}^{\infty} d\omega\, \omega\, \Delta\rho_{0i;0j}(\omega, \boldsymbol{q}, T) &= q_i q_j (e + p - \tfrac{8}{33}(e - 3p)) \\
&+ (\delta_{ij}\boldsymbol{q}^2 - q_i q_j)(\tfrac{1}{2}(e+p) + \tfrac{5}{33}(e - 3p)).
\end{aligned}
$$

There is an additional sum rule in the sound channel,

$$
-q_i q_j \int_{-\infty}^{\infty} \frac{d\omega}{\omega} \Delta\rho_{0i;0j}(\omega, \boldsymbol{q}, T) = \boldsymbol{q}^2 (e + p).
$$

Valid for any $\boldsymbol{q}$.

Axial vector channel

Chiral WI:

$$k_\mu k_\nu \langle A^b_\nu(0) \int d^4x\, e^{ikx} A^a_\mu(x)\rangle = -2m\delta^{ab}\langle\bar\psi\psi\rangle + 4m^2\langle P^b(0)\int d^4x\, e^{ikx} P^a(x)\rangle$$

Thus, the correlators of A^a_i can be expressed in terms of the A^a_0 and the P^a correlators.

In particular, the following sum rules hold:

$$\begin{aligned}
\int_{-\infty}^{\infty} d\omega\, \omega\, \Delta\rho^A_{00}(\omega, \boldsymbol{k}, T) &= -2m\langle\bar\psi\psi\rangle_{T-0}, \\
4m^2 \int_{-\infty}^{\infty} \frac{d\omega}{\omega}\, \Delta\rho_{PP}(\omega, \boldsymbol{k}=0, T) &= 2m\langle\bar\psi\psi\rangle_{T-0}.
\end{aligned}$$

$A^a_\mu(x) = \bar\psi\gamma_\mu\gamma_5\frac{\tau^a}{2}\psi(x), \qquad P^a(x) = \bar\psi(x)\gamma_5\frac{\tau^a}{2}\psi(x), \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$

‘Static’ second-order hydrodynamic coefficients

Constitutive equation for j_μ (in rest frame of fluid, [Hong, Teaney ’10]):

$$\boldsymbol{J} = \sigma(1 - \tau_J \partial_t)\boldsymbol{E} + \kappa_B \nabla \times \boldsymbol{B}.$$

Constitutive equation for $T_{\mu\nu}$ ($\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ [Baier et al. ’07]),

$$\begin{aligned} T^{\mu\nu} &= 2\eta \nabla^{\langle\mu} u^{\nu\rangle} + \cdots + \kappa(R^{\langle\mu\nu\rangle} - 2u_\alpha u_\beta R^{\alpha\langle\mu\nu\rangle\beta}) + \ldots, \\ A^{\langle\mu\nu\rangle} &= \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(A_{\alpha\beta} + A_{\beta\alpha}) - \frac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}A_{\alpha\beta}, \end{aligned}$$

Kubo formulae: [Hong, Teaney ’10; Moore, Sohrabi ’10]

$$\kappa_B = \kappa_t = -\left.\frac{1}{2}\partial^2_{q_3}\Pi_{11}(q_3\boldsymbol{e}_3)\right|_{q_3=0}, \qquad \kappa = -\partial^2_{q_3} G^{12,12}(q_3\boldsymbol{e}_3)|_{q_3=0}$$

- κ: SU(3) gauge theory calculation by O. Philipsen and Ch. Schäfer
- κ_B: $N_\mathrm{f} = 2$ calculation (A. Francis & HM).

Interpretation of κ_t and κ_ℓ A. Francis, HM

Two static leptons in the QED or QCD plasma: free energy given at long distance by

$$V_C(R) = e^2(1+e^2\kappa_l)\,\frac{Q_1 Q_2\, e^{-m_{\rm el} r}}{4\pi r}, \qquad m_{\rm el}^2 = e^2\chi_s.$$

Main effect is Debye screening.

Consider the setup used to define the ampère unit. In the plasma, the Ampère force (defined by $\boldsymbol{F}_A(\boldsymbol{R}_\perp) = -\nabla(\text{ free energy})$) is given at long distance by

$$\frac{1}{L_3} F_A(\boldsymbol{R}_\perp) = -e^2(1+e^2\kappa_t)\cdot\frac{I_1 I_2 \boldsymbol{R}_\perp}{2\pi R_\perp^2}$$

Since $\kappa_t > 0$, the Ampère force is enhanced compared to the force measured in vacuum.

Subtraction of $T=0$ correlator in defn of κ's means renormalizing bare charge e_0.

One-loop expressions

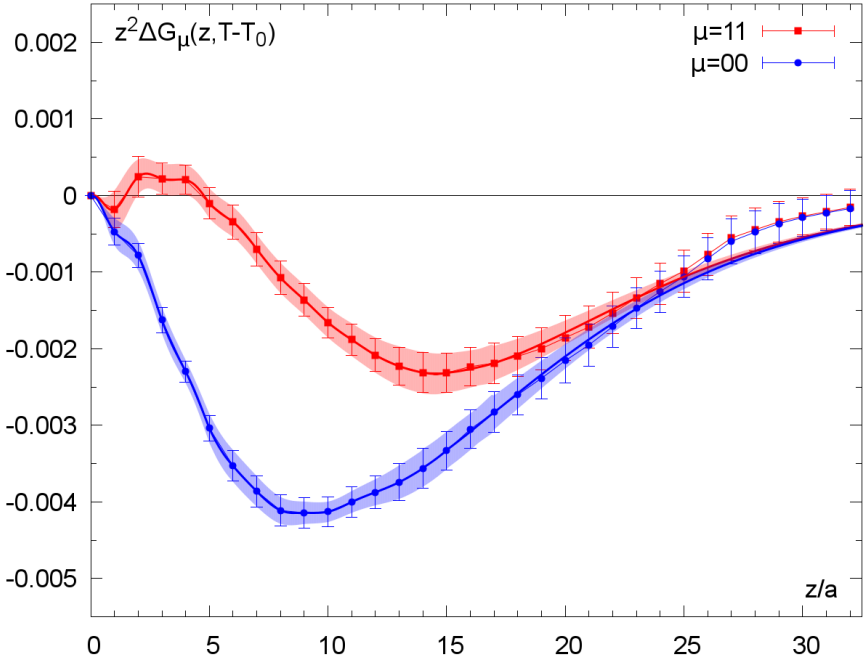
$$
\begin{aligned}
\chi_s &= \frac{2}{\pi^2}\int_0^\infty \frac{dp}{E_{\boldsymbol{p}}}\, n_F(E_{\boldsymbol{p}})\,(p^2+E_{\boldsymbol{p}}^2) = -\frac{2m^2}{\pi^2}\sum_{n\geq 1}(-1)^n\, K_2(n\,\beta m),\\
\kappa_t &= \frac{1}{3\pi^2}\int_0^\infty \frac{dp}{E_{\boldsymbol{p}}}\, n_F(E_{\boldsymbol{p}}) = -\frac{1}{3\pi^2}\sum_{n=1}^{\infty}(-1)^n\, K_0(n\,\beta m),\\
\kappa_\ell - \kappa_t &= -\frac{1}{6\pi^2}\int_0^\infty dp\, n_F'(E_{\boldsymbol{p}}) = -\frac{\beta m}{6\pi^2}\sum_{n=1}^{\infty}(-1)^n\, n\, K_1(n\,\beta m).
\end{aligned}
$$

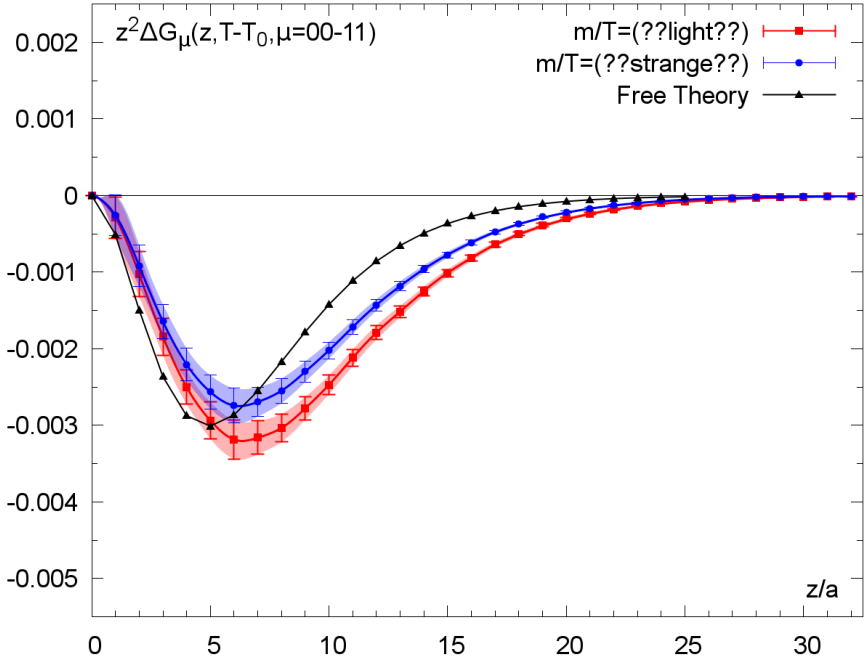
In the massless limit, ($b \equiv \frac{m}{\pi T}$):

$$
\begin{aligned}
\chi_s &= T^2/3,\\
\kappa_t &= \frac{1}{6\pi^2}\left(\log\left(\frac{\pi T}{m}\right) - \gamma_{\mathrm{E}}\right),\\
\kappa_\ell - \kappa_t &= \frac{1}{12\pi^2}.
\end{aligned}
$$

Lattice results for κ_t and κ_ℓ

- Area under these curves:





- $N_\mathrm{f} = 2,\ \ T = 255\mathrm{MeV},\ \ m_\pi = 270\mathrm{MeV},\ \ 16 \times 64^3$
- $\kappa_t = \int_0^\infty dx_3\ x_3^2 \langle \tilde{j}_1(x) j_1(0) \rangle |_0^T, \qquad \kappa_\ell = \int_0^\infty dx_3\ x_3^2 \langle \tilde{j}_0(x) j_0(0) \rangle |_0^T$

Conclusion

- spectral functions at zero temperature: many opportunities for lattice QCD at small ω
- sum rules help constrain thermal spectral functions
- quality of Euclidean data = its ability to exclude models of the spectral function.

Announcement

Jets, particle production and transport properties in collider and cosmological environments

Mainz Institute for Theoretical Physics (MITP)
July 28 - August 8, 2014.

Dietrich Bodeker, Nora Brambilla, Harvey Meyer, Guy Moore, Frank Steffen, Antonio Vairo

Backup Slides

Pion quasiparticle below T_c

Dispersion relation of the pion quasiparticle [Son-Stephanov '02]

$$\begin{aligned} \omega_{\boldsymbol{p}}^2 &= u(T)^2(\boldsymbol{p}^2 + m(T)^2), \\ m(T) &= \text{pseudoscalar screening mass}, \qquad u(T)^2 = \frac{f(T)^2}{\chi_A}. \end{aligned}$$

On the choice of operators

- flux associated with a conserved charge is not uniquely defined
- e.g. $j_\mu(x) \to j_\mu(x) + i\lambda\partial_\nu(\bar{\psi}(x)\sigma_{\mu\nu}\psi(x))$, $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu]$, does not affect the conserved charges nor the conservation equation $\partial_\mu j_\mu = 0$.
- derivation of the Kubo formula shows that any conserved vector current must exhibit the diffusion pole, with same residue to leading order in $\boldsymbol{k}$,

$$\frac{\pi\rho^{nn}(\omega, \boldsymbol{k})}{\omega} = \frac{\chi_s^{nn}(\beta) D \boldsymbol{k}^2}{(D\boldsymbol{k}^2)^2 + \omega^2} + \dots$$

- even if $q(x)$ is not conserved, but has same quantum numbers as the charge density $j_0(x) \equiv n(x)$, expect its retarded correlator to exhibit diffusion pole
- assuming that global equilibrium is reached via a partition of fluid cells that are in local equilibrium,

$$\frac{\pi\rho^{qq}(\omega, \boldsymbol{k})}{\omega} = \left(\frac{\chi_s^{nq}(\beta)}{\chi_s^{nn}(\beta)}\right)^2 \frac{\pi\rho^{nn}(\omega, \boldsymbol{k})}{\omega} + \dots,$$

- smooth operator $q(x)$ would have the advantage of not having an ultraviolet tail (use Wilson flow Lüscher '10).