Dirac Spectrum of a 2d Theory with $\Sigma = 0$

I. Chiral symmetry and low lying Dirac eigenvalues

II. The $N_f = 2$ Schwinger model

III. Decorrelation of small Dirac eigenvalues?
   - Leading EVs in fixed volume
   - $\langle \lambda_n \rangle$ in different volumes
   - Unfolded level spacing density

IV. Mass anomalous dimension

V. Conclusions

I. Chiral Symmetry and low lying Dirac eigenvalues

Two scenarios for the chiral condensate $\Sigma$ at fermion mass $m \to 0$:

- **Chiral sym. breaking** : $\Sigma(m = 0) > 0$
  
  e.g. low-T QCD (SSB), $N_f = 1$ Schwinger model (axial anomaly)

- **Theories with** $\Sigma(m = 0) = 0$
  
  IR conformal theories:
  high-T QCD; multi-flavor QCD : $N_f = 8, 12$ ??? (review [1])

Schwinger model with $N_f \geq 2$. In continuum and large volume:

$$\Sigma(m) \propto m^{1/\delta}, \quad \delta = \frac{N_f + 1}{N_f - 1} \quad [2].$$
Typical behavior if $\Sigma(m = 0) = 0$:

Power low for density $\rho$ of small Dirac eigenvalues $\lambda$

$$\rho(\lambda) = cV|\lambda|^\alpha, \quad \alpha = 1/\delta$$

(no Banks-Casher plateau). For high-$T$ : $V = \text{spatial volume}$.

**EV decorrelation** ↔ **Poisson statistics** implies

$$\rho_n(\lambda) = \frac{(cV)^n}{(n-1)!} \frac{\lambda^{n(\alpha+1)-1}}{(\alpha+1)^{n-1}} \exp\left(-\frac{cV}{\alpha+1}\lambda^{\alpha+1}\right) \quad (1)$$

Iterated formula by T.G. Kovács [3].
II. Schwinger model with $N_f = 2$ light fermions

$\delta$ depends on the parameter $l$ [4] (with $L = \sqrt{V}$, $g$ : gauge coupling)

\[
l := \frac{m}{\pi^{1/4} \sqrt{2L^3g}} \begin{cases} 
\gg 1 & \delta = 3 \ [1] \\
\ll 1 & \delta = 1 = \alpha \ [\text{like free fermion, } \rho \propto \lambda^{d-1}] 
\end{cases}
\]

- Simulations on $L \times L$ lattices, with dynamical chiral fermions (overlap hypercube [5]) of degenerate mass $m$.

$\beta = 1/g^2 = 5 \rightarrow \langle \text{plaquette} \rangle \simeq 0.9$ (close to continuum)

<table>
<thead>
<tr>
<th>$m$</th>
<th>$l$</th>
<th>$L = 16$</th>
<th>$L = 32$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>0.455</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.06</td>
<td>2.728</td>
<td></td>
<td>7.715</td>
</tr>
</tbody>
</table>

Table 1: Parameter $l$ for our fermion masses $m$ and lattices sizes $L$.

Eigenvalues $\lambda_n$ are mapped from the Ginsparg-Wilson circle onto a line.
III. Decorrelation of small Dirac eigenvalues ?

Fits of cumulative EV densities to Kovács distribution (1), $R_n(\lambda) = \int_0^\lambda d\lambda' \rho_n(\lambda')$, at top. charge $\nu$. Free parameters: $c, \alpha$ (cf. p.3)
Fits of the mean eigenvalues \( \langle \lambda_n \rangle \) to values according to the Kovács distribution in volumes \( 16^2 \ldots 32^2 \).

On the left: **fit okay (bold line)**, but is inconsistent with the parameters \( c, \alpha \) of the cumulative density fits (p.5, fine lines).

Individual fits all work, but the parameters are totally inconsistent: \( \alpha \) varies from 0.58(3) to 8.08(5), \( c \) even from 0.09(3) to 2.7(2) \( \times 10^4 \).
Cumulative density of unfolded level spacings $s$ for the lowest two EVs:

As with the entire spectrum [6], also the lowest EVs are close to the distribution of the Chiral Unitary Ensemble, $32(s/\pi)^2 \exp(-4s^2/\pi)$ [7], but far from the Poisson distribution $\exp(-s)$.

This is in contrast to high-T $(2 + 1)$ flavor QCD [8].
IV. Mass anomalous dimension

As suggested in Ref. [9], we employ the mode number [10]

$$\nu_{\text{mode}}(\lambda) = V \int_{-\lambda}^{\lambda} d\lambda' \, \rho(\lambda') = \frac{2cV^2}{\alpha + 1} \lambda^{\alpha+1}$$

as a tool to evaluate the mass anomalous dimension

$$\gamma_m(\lambda) = \frac{d}{\alpha(\lambda) + 1} - 1.$$ 

We are most interested in the IR limit,

$$\gamma_m^* = \lim_{\lambda \to 0} \gamma_m(\lambda).$$
Consideration of $\lambda = 0.6 \ldots 2$ suggests, for both masses, very similar IR extrapolations:

- $m = 0.01 : \gamma^*_{m} = 0.065(5)$ ,
- $m = 0.06 : \gamma^*_{m} = 0.063(7)$ .

Extrapolations $m \to 0$ and $L \to \infty$ do not seem relevant here, and lattice artifacts are small anyhow (cf. p.4).

However: fit based on EV at higher energies $\Rightarrow$ lower $\gamma^*_{m}$.
V. Conclusions

Hypothesis: in theories with $\Sigma = 0$, small Dirac EV are decorrelated.

Confirmed for high-T YM theories [3, 8], but not for $N_f = 2$ Schwinger model:
1) Fits work, but require inconsistent parameters
2) Unfolded level spacings incompatible with Poisson distribution.

Modified Hypothesis: EV decorrelation only for high-T IR conformal theories, with $1/T = $ localization scale for small Dirac EVs [11].

Mass anomalous dimension

Apparently reliable results for IR extrapolated value $\gamma^*_m$, but: value decreases for fit based on larger EVs; lowest EVs suggest $\alpha = 3/5 \Rightarrow \gamma^*_m = 1/4$ [6].

Dependence on $l \propto mL^{3/2}g^{1/2}$ [4]: $l \ll 1 \Rightarrow \gamma^*_m = 0$, $l \gg 1 \Rightarrow \gamma^*_m = 1/2$.

$\Rightarrow \gamma^*_m$ is tricky, as in multi-flavor QCD, see e.g. [1, 9, 12].
References


*We thank Poul Damgaard, Philippe de Forcrand, James Hetrick, Christian Hoelbling, Tamas Kovács and Andrei Smilga for helpful comments.*